Atividade Sólido de Einstein.

Número de microestados: $\Omega(E,N) = \dfrac{\left(\frac{E}{\hbar\omega} + \frac{N}{2} - 1\right)!}{\left(\frac{E}{\hbar\omega} - \frac{N}{2}\right)!\,(N-1)!}$

(a) Determinar a entropia

$s(u) = \lim_{\substack{N\to\infty \\ E/N = u}} \frac{k_B}{N}\ln\Omega$, usando $\ln N! = N\ln N - N$

mostrar principais passagens

$$\frac{s(u)}{k_B} = \left(\frac{u}{\hbar\omega} + \frac{1}{2}\right)\ln\left(\frac{u}{\hbar\omega} + \frac{1}{2}\right) - \left(\frac{u}{\hbar\omega} - \frac{1}{2}\right)\ln\left(\frac{u}{\hbar\omega} - \frac{1}{2}\right) \quad //$$

(b) $ds = \frac{1}{T}du$, $\frac{\partial s}{\partial u} = \frac{1}{T}$

$$\Rightarrow \frac{1}{T} = k_B\left[\frac{1}{\hbar\omega}\ln\left(\frac{u}{\hbar\omega} + \frac{1}{2}\right) + \cancel{\frac{1}{\hbar\omega}} - \frac{1}{\hbar\omega}\ln\left(\frac{u}{\hbar\omega} - \frac{1}{2}\right) - \cancel{\frac{1}{\hbar\omega}}\right]$$

$$\Rightarrow \frac{1}{T} = \frac{k_B}{\hbar\omega}\ln\left(\frac{u}{\hbar\omega} + \frac{1}{2}\right) - \frac{k_B}{\hbar\omega}\ln\left(\frac{u}{\hbar\omega} - \frac{1}{2}\right) = \frac{k_B}{\hbar\omega}\ln\left[\frac{u/\hbar\omega + 1/2}{u/\hbar\omega - 1/2}\right]$$

Resolvendo para u: $\exp\left(\frac{\hbar\omega}{k_B T}\right) = \dfrac{u/\hbar\omega + 1/2}{u/\hbar\omega - 1/2}$

$$\frac{u}{\hbar\omega}\left(\exp(\beta\hbar\omega) - 1\right) - \frac{1}{2}\exp(\beta\hbar\omega) = \frac{1}{2}$$

$$\frac{u}{\hbar\omega} = \frac{\frac{1}{2}\left(1 + \exp(\beta\hbar\omega)\right)}{\exp(\beta\hbar\omega) - 1} \Rightarrow u = \frac{\hbar\omega}{2}\,\frac{1 + \exp(\beta\hbar\omega)}{\exp(\beta\hbar\omega) - 1}$$

Limite de baixas temperaturas:

$$\lim_{T\to 0^+} u(T) = \frac{\hbar\omega}{2}\lim_{T\to 0^+} \underbrace{\frac{\cancel{\exp(\hbar\omega/k_BT)}}{\cancel{\exp(\hbar\omega/k_BT)}}}_{\to 1} \cdot \frac{1 + \overbrace{1/\cancel{\exp(\hbar\omega/k_BT)}}^{\to 0}}{1 - \underbrace{1/\cancel{\exp(\hbar\omega/k_BT)}}_{\to 0}} = \frac{\hbar\omega}{2}$$

Energia do estado fundamental.

Limite de altas temperaturas $k_B T \gg \hbar\omega$, $\frac{\hbar\omega}{k_B T} = x$

$$u(T) = \hbar\omega\underbrace{\left(\frac{1}{2} + \frac{1}{e^x - 1}\right)}, \quad e^x = 1 + x + \theta(x^2)$$

$$u(T) = \hbar\omega\left(\frac{1}{2} + \frac{1}{x}\right), \text{ mas } x \ll 1 \Rightarrow \frac{1}{x} \gg \frac{1}{2}$$

$$u(T) \simeq \frac{\hbar\omega}{\hbar\omega/k_B T} \Rightarrow \boxed{u(T) = k_B T}$$

$\mathcal{H} = \frac{p^2}{2m} + kx^2$, então, pelo teorema de equipartição

da energia temos $u = 2 \cdot \frac{1}{2} k_B T = k_B T$.

(c) Calor específico $c = \frac{\Delta Q}{\Delta T} = T\frac{\partial s}{\partial T} = \cancel{T} \cdot \frac{1}{\cancel{T}}\frac{\partial u}{\partial T}$

$$c = \frac{\partial u}{\partial T} = \frac{d}{dT}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1}\right]$$

$$c = \frac{-\hbar\omega}{\left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^2}\left(\frac{-\hbar\omega}{k_B T^2}\right)\exp\left(\frac{\hbar\omega}{k_B T}\right) = \frac{(\hbar\omega)^2}{k_B T^2}\frac{\exp\left(\hbar\omega/k_B T\right)}{\left[\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1\right]^2}$$

$$\text{Limite } T \to 0 \quad c \propto \frac{1}{T^2}\underbrace{\exp\left(-\frac{\hbar\omega}{k_B T}\right)}_{\text{decaimento exponencial}} \to 0$$