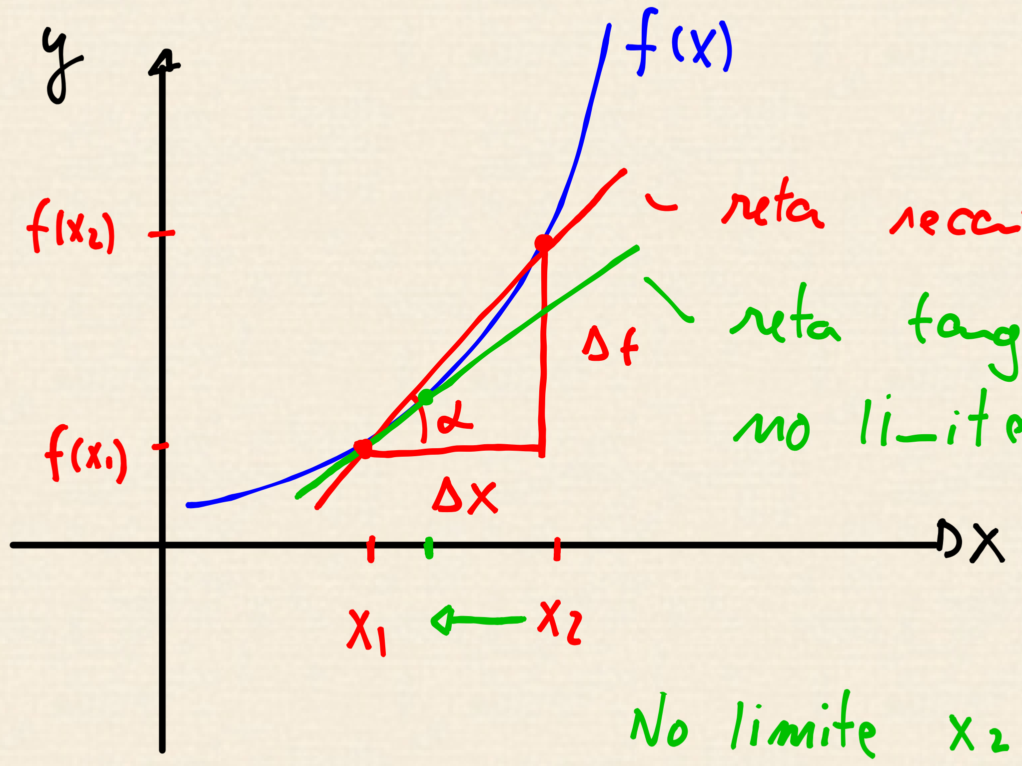
VIDEO AULA 5

# DERIVADAS




reta secante $tg(\alpha) = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

reta tangente no limite $x_2 \to x_1$.

No limite $x_2 \to x_1$, temos a reta tangente ao ponto $(x_1, f(x_1))$.

O ângulo da reta tangente é dado por

$$tg(\theta) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

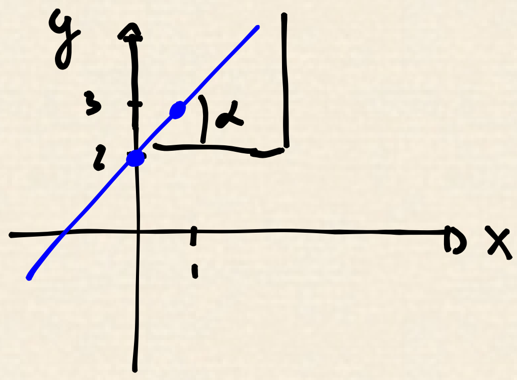
$x_2 = x_1 + h$

Em geral: $tg(\theta) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Derivada de $f(x)$ com relação a $x$:

$$\boxed{\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

**Ex. 1** Ângulo de $f(x) = x+2$ com a horizatol.




$$tg(\alpha) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x} + h + \cancel{2} - \cancel{x} - \cancel{2}}{h} = \lim_{h\to 0} 1 = 1 \;//$$

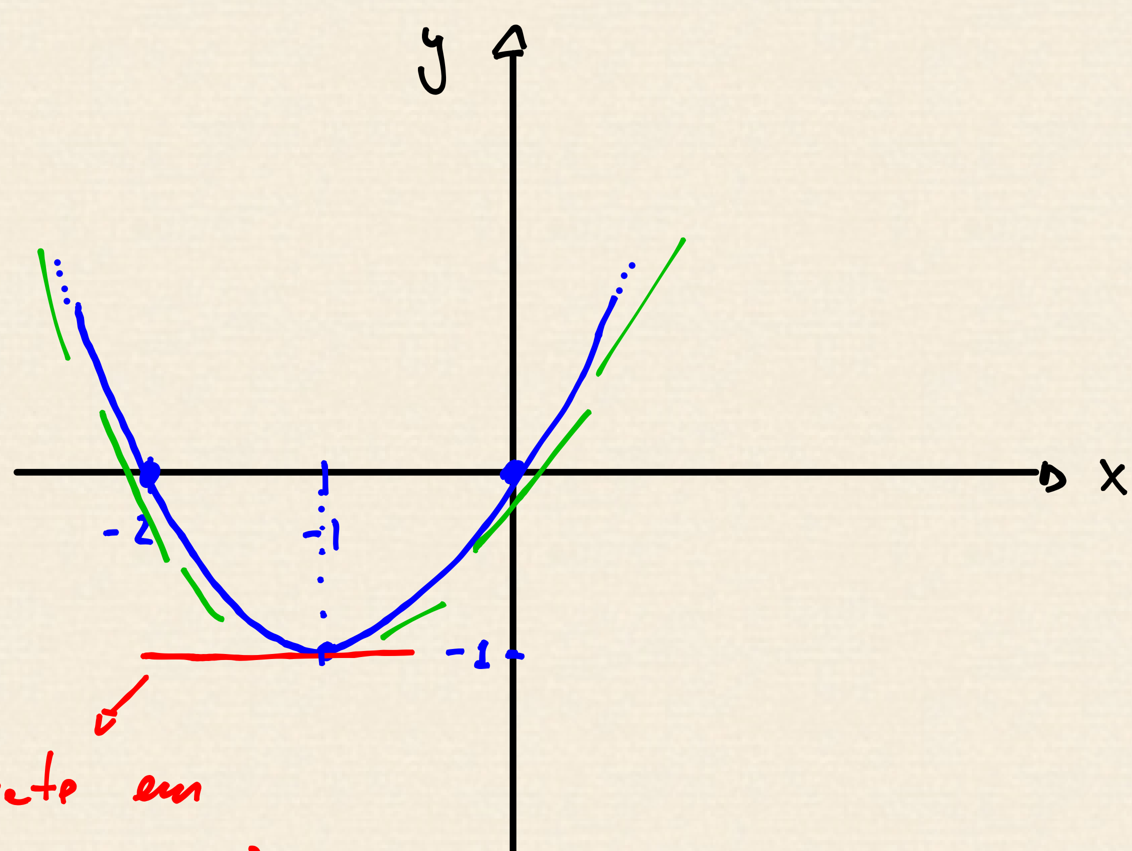
$$\Rightarrow \boxed{\alpha = 45^\circ}$$

**Ex. 2** $f(x) = x^2 + 2X$, ângulo da reta tagente em $X = -1$.

$$tg(\alpha) = \lim_{h\to 0} \frac{(x+h)^2 + 2(\cancel{x}+h) - x^2 - \cancel{2x}}{h} = \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 + 2h - \cancel{x^2}}{h}$$

$$tg(\alpha) = \lim_{h\to 0} \left[ 2x + h + 2 \right] = 2X + 2$$

$$tg(\alpha)\Big|_{X=-1} = 2(-1) + 2 = 0 \;// \qquad \alpha = 0^\circ \;//$$




reta tagente em

$X = -1, \quad \alpha = 0^\circ \;//$

$$f(x) = 0 \Rightarrow x^2 + 2x = 0 \;\to\; x_1 = 0, \quad x_2 = -2$$

$$f(x = -1) = -1$$

# Derivada como taxa de variação:

Posição : $x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$

Taxa de variação da posição em relação ao tempo:

Velocidade $v(t) = \frac{d\,x(t)}{dt} = \dot{x}(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$.

Taxa de variação da velocidade em relação ao tempo:

Aceleração $a(t) = \frac{d\,v(t)}{dt} = \dot{v}(t) = \frac{d^2}{dt^2} x(t) = \ddot{x}(t)$.

## Existência da Derivada

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Se o limite não existir para $c \in D_f$, $f$ é não derivável em $x = c$.

Ex. $f(x) = |x-1|$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

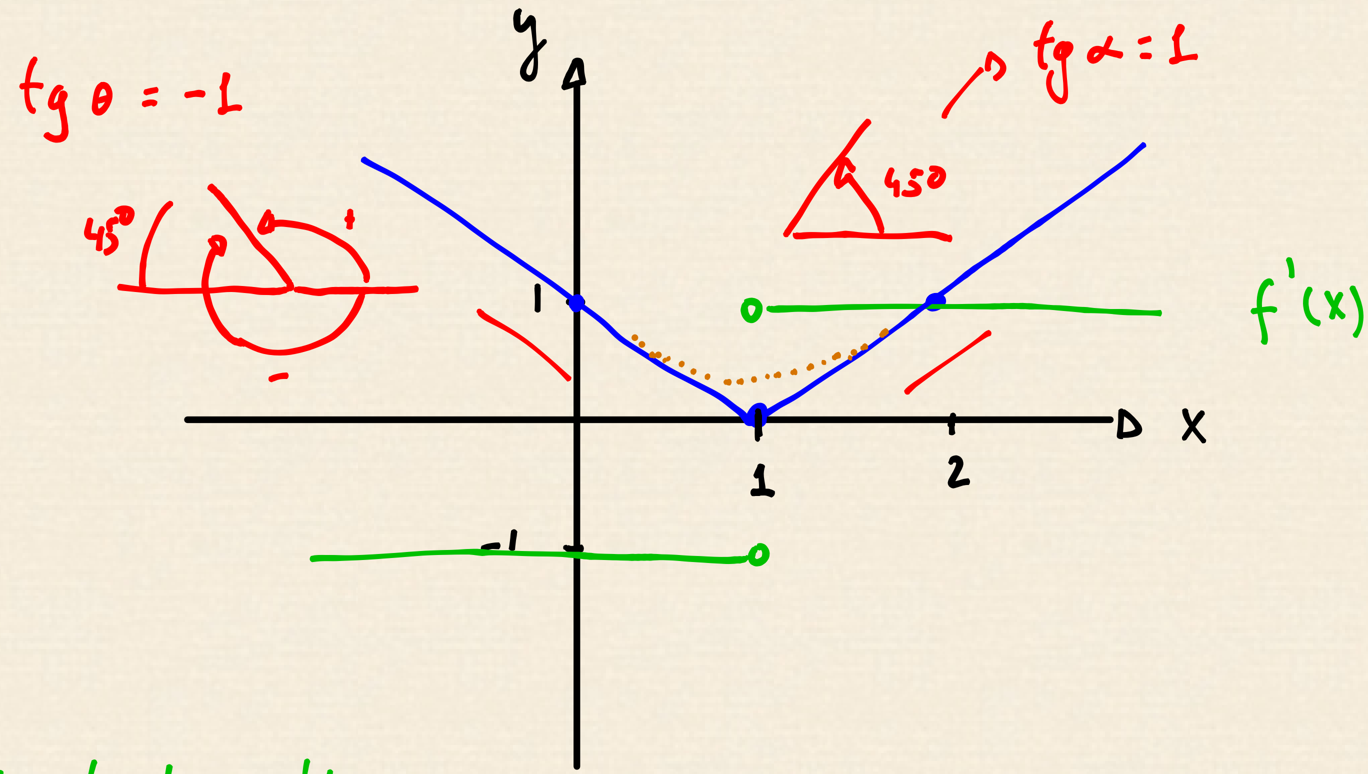
$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \qquad f(1) = 0$$

$$= \lim_{h \to 0^+} \frac{\cancel{1}+h-\cancel{1} - \overbrace{(\cancel{1}-\cancel{1})}^{f(1)=0}}{h} = 1$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{-(\cancel{1}+h-\cancel{1}) - 0}{h} = -1$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \neq \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \Rightarrow \underline{\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \nexists}$$

Então $f$ é não derivável em $x = 1$ ($f'(1) \nexists$).




$f'(x)$ é descontínua em $x = 1$.

$f(x)$ é contínua.

$$\boxed{\text{TEO.: Uma função derivável é contínua.}}$$

Continuidade: $\lim\limits_{x \to c} f(x) = f(c)$

$$f(x) - f(c) = \frac{f(x) - f(c)}{(x - c)} \cdot (x - c)$$

$$\lim_{x \to c} \left( f(x) - f(c) \right) = \underbrace{\lim_{x \to c} \left[ \frac{f(x) - f(c)}{x - c} \right]}_{\text{Se } f'(x)\ \exists} \cdot \cancel{\lim_{x \to c} (x - c)}^{\to 0} = 0$$

$$\Rightarrow \lim_{x \to c} \left( f(x) - f(c) \right) = 0$$

$$\lim_{x \to c} f(x) = f(c) \Rightarrow f \text{ é contínua.}$$

# Regras de Derivação

① $f(x) = c\,g(x)$ , $f'(x) = c\,g'(x)$

② $f(x) = g(x) + h(x)$ , $f'(x) = g'(x) + h'(x)$

A derivada é um operador linear.

③ $f(x) = g(x)h(x)$ , $f'(x) = g'(x)h(x) + g(x)h'(x)$

④ $f(x) = \dfrac{g(x)}{h(x)}$ , $f'(x) = \dfrac{g'(x)h(x) - g(x)h'(x)}{h^2(x)}$

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# Regras de derivação

① função constante : $f(x) = C = \text{const.}$

$$f'(x) = 0 \quad , \quad \frac{dC}{dx} = 0$$

② função potência: $f(x) = x^n \ , \ n \in \mathbb{Q}^*$

$$f'(x) = n\,x^{n-1} \quad , \quad \frac{d}{dx} x^n = n\,x^{n-1}.$$

Lembrar dos limites

$$\frac{d}{dt}(t^2) = \lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} = 2t \quad , \quad \lim_{h\to 0} \frac{(t+h)^3 - t^3}{h} = 3t^2$$

$$\frac{(x+h)^n - x^n}{h} = \frac{x^n + n\,x^{n-1}h + \dots + h^n - x^n}{h} \underset{\text{"proporcional"}}{\propto} \frac{n x^{n-1} \cancel{h} + \mathcal{O}(h)}{\cancel{h}}$$

③ função polinomial : $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x^1 + a_0 x^0$

$$n \in \mathbb{N}$$

$$\frac{d}{dx}(a_n x^n) = a_n \frac{dx^n}{dx} = a_n\, n\, x^{n-1}$$

$$f'(x) = a_n\, n\, x^{n-1} + a_{n-1}(n-1)\, x^{n-2} + \cdots + 1 \cdot a_1 \cdot x^0 + 0$$

Ex.1 $f(x) = x^3 + 2x - 1$

$$f'(x) = 3x^2 + 2$$

Ex.2 $f(x) = (3x^4 + 2x^2)\,x^{-2} = 3x^2 + 2 \ , \ x \neq 0$

$$f'(x) = \frac{d}{dx}(3x^4 + 2x^2) \cdot x^{-2} + (3x^4 + 2x^2)\frac{d}{dx} x^{-2}$$

$$f'(x) = \frac{d}{dx}(3x^4+2x^2)\cdot x^{-2} + (3x^4+2x^2)\frac{dx^{-2}}{dx}$$

$$f'(x) = (3\cdot 4x^3 + 2\cdot 2x)\,x^{-2} + (3x^4+2x^2)(-2x^{-3})$$

$$f'(x) = 12x + \cancel{4x^{-1}} - 6x - \cancel{4x^{-1}} = 6x$$

$$f(x) = 3x^2 + 2$$

$$f'(x) = 6x$$

Ex.3 $f(x) = \dfrac{x^3 - 2x^2}{3x-8}$

Regra do quociente.

$$f'(x) = \frac{(x^3-2x^2)'(3x-8) - (x^3-2x^2)(3x-8)'}{(3x-8)^2}$$

$$f'(x) = \frac{(3x^2-4x)(3x-8) - (x^3-2x^2)\,3}{(3x-8)^2}$$

$$f'(x) = (9x^3 - 24x^2 - 12x^2 + 32x - 3x^3 + 6x^2)/(3x-8)^2$$

$$f'(x) = \frac{6x^3 - 30x^2 + 32x}{(3x-8)^2}$$

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# Regra da Cadeia

$$y = g(u) \quad , \quad u = f(x) \quad , \quad y = g(f(x))$$

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Ex. 1 $\quad f(x) = (x^3 + 2x - 1)^3$

$$u(x) = x^3 + 2x - 1 \quad , \quad f = u_{(x)}^3$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{d}{du}u^3 \cdot \frac{d}{dx}(x^3 + 2x - 1)$$

$$\Rightarrow \frac{df}{dx} = 3u^2(3x^2 + 2) \Rightarrow \frac{df}{dx} = 3(x^3 + 2x - 1)^2(3x^2 + 2)$$

Ex. 2 $\quad f(x) = \frac{g(x)}{h(x)} = g(x) \cdot h_{(x)}^{-1}$

$$\frac{df(x)}{dx} = \frac{dg}{dx}h^{-1} + g\frac{dh^{-1}}{dx} \quad , \quad \frac{d(h(x))^{-1}}{dx} = \frac{dh^{-1}}{dh}\frac{dh}{dx} = -h^{-2}\frac{dh}{dx}$$

$$\frac{d}{dx}f(x) = \frac{dg}{dx}h^{-1} - g\,h^{-2}\frac{dh}{dx} = \frac{\frac{dg}{dx}h - g\,dh/dx}{h^2}$$

Ex.3 $f(x) = \dfrac{3x+2}{2x+1} = (3x+2)(2x+1)^{-1}$

$$\frac{df}{dx} = 3(2x+1)^{-1} + (3x+2)\frac{d}{dx}(2x+1)^{-1}$$

$$\frac{d}{dx}(2x+1)^{-1} = \frac{d}{dx}u_{(x)}^{-1} = \frac{du^{-1}}{du}\frac{du}{dx} = -u^{-2}\cdot 2 \quad , \quad u = 2x+1$$

$$\Rightarrow \frac{df}{dx} = 3(2x+1)^{-1} - 2(3x+2)(2x+1)^{-2} \ //$$

Ex.4 $f(x) = \left(\dfrac{3x+2}{2x+1}\right)^5$

$$u = \frac{3x+2}{2x+1} \quad , \quad \frac{df}{dx} = \frac{d}{dx}u^5 = \frac{du^5}{du}\frac{du}{dx} = 5u^4\frac{du}{dx}$$

$$\frac{df}{dx} = 5\left(\frac{3x+2}{2x+1}\right)^4\left[3(2x+1)^{-1} - 2(3x+2)(2x+1)^{-2}\right] \ //$$

Ex.5 $f(x) = 5\sqrt{x^2+3}$

$$u = x^2+3 \quad , \quad \frac{df}{dx} = \frac{d}{dx}5\sqrt{u} = 5\frac{du^{1/2}}{du}\frac{du}{dx}$$

$$\Rightarrow \frac{df}{dx} = 5\cdot\frac{1}{\cancel{2}}u^{-1/2}\,\cancel{2}x \Rightarrow \frac{df}{dx} = \frac{5x}{\sqrt{x^2+3}} \ //$$

Ex.6 $f(x) = \dfrac{1}{(1-x^2)^{3/2}} = (1-x^2)^{-3/2}$

$u = 1-x^2$ , $\dfrac{df}{dx} = \dfrac{d}{dx} u^{-3/2} = \dfrac{du^{-3/2}}{du} \dfrac{du}{dx}$

$\Rightarrow \dfrac{df}{dx} = -\dfrac{3}{\cancel{2}} u^{-5/2} (-\cancel{2}x) = \dfrac{3x}{(1-x^2)^{5/2}}$ //

Exercícios da Lista, seção 6 e 7.

SEC. 6, (1) Reta tangente de $f(x) = 2x^2 - 4x + 5$

em $x = 1$.

$$f'(1) = tg(\theta) = \lim_{h \to 0} \left. \frac{f(x+h) - f(x)}{h} \right|_{x=1} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$tg(\theta) = \lim_{h \to 0} \frac{2(1+h)^2 - 4(1+h) + 5 - [2 - 4 + 5]}{h}$$

$$tg(\theta) = \lim_{h \to 0} \frac{\cancel{2} + \cancel{4h} + 2h^2 \cancel{- 4} - \cancel{4h} + \cancel{5} - \cancel{2} + \cancel{4} - \cancel{5}}{h}$$

$$tg(\theta) = \lim_{h \to 0} \frac{2h^2}{h} = \lim_{h \to 0} 2h = 0$$

$$tg(\theta) = 0 \Rightarrow \theta = 0^\circ \;//$$

Ângulo da reta tangente a f em $x = 1$.

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$$f(x) = 2x^2 - 4x + 5$$

$$f'(x) = 4x - 4, \quad f'(1) = 0 = tg(\theta) \to \theta = 0^\circ$$

③ $f(x) = \begin{cases} 3x-1, & x<2 \\ 7-x, & x \geq 2 \end{cases}$

$$f'(x=2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h}$$

$$\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0^-} \frac{3(2+h)-1-(7-2)}{h}$$

$$= \lim_{h \to 0^-} \frac{\cancel{6}+3h-\cancel{1}-\cancel{5}}{h} = \lim_{h \to 0^-} \frac{3\cancel{h}}{\cancel{h}} = 3$$

$$\lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h \to 0^+} \frac{\cancel{7}-(\cancel{2}+h)-(\cancel{7}-\cancel{2})}{h}$$

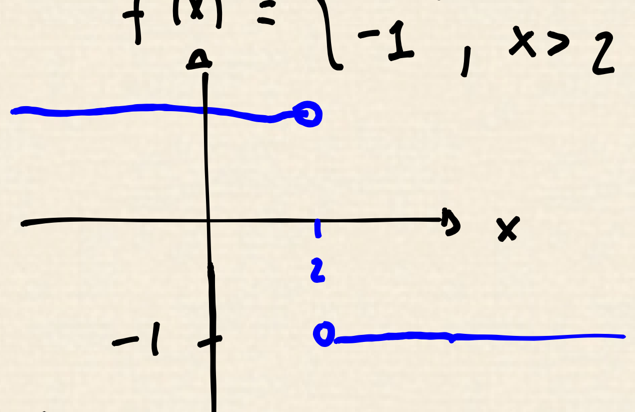
$$= \lim_{h \to 0^+} \frac{-h}{h} = -1$$

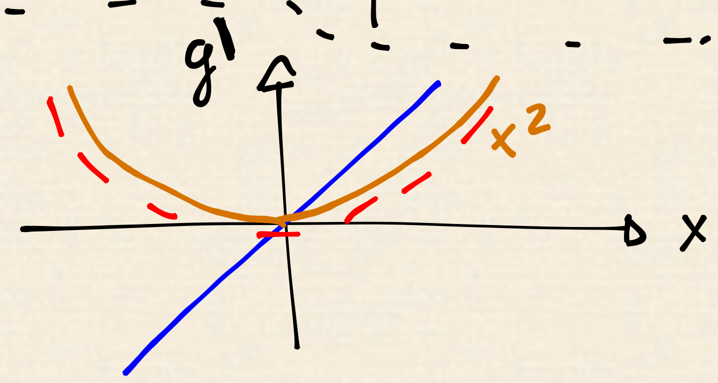
$$\lim_{h \to 0^-} \frac{f(2+h)-f(2)}{h} \neq \lim_{h \to 0^+} \frac{f(2+h)-f(2)}{h}$$

$$\Rightarrow f'(2) = \lim_{h \to 0} \frac{f(2+h)-f(2)}{h} \nexists$$

Então $f$ é não derivável em $x=2$.

$$f'(x) = \begin{cases} 3, & x<2 \\ -1, & x>2 \end{cases}$$




$g = x^2$

$g' = 2x$

SEC. 7

(3) $f(x) = (x^3 - x^2 + 3)x^{-5}$

Pela regra do produto: $f'(x) = (x^3 - x^2 + 3)' x^{-5} + (x^3 - x^2 + 3)(x^{-5})'$

$$f'(x) = (3x^2 - 2x)x^{-5} + (x^3 - x^2 + 3)(-5) \cdot x^{-6}$$

$$f'(x) = \frac{3x^2 - 2x}{x^5} - \frac{5x^3 - 5x^2 + 15}{x^6}$$

$$f'(x) = \frac{3x^3 - 2x^2}{x^6} - \frac{(5x^3 - 5x^2 + 15)}{x^6}$$

$$f'(x) = \frac{-2x^3 + 3x^2 - 15}{x^6} = -2x^{-3} + 3x^{-4} - 15x^{-6} \quad //$$

Outra maneira: $f(x) = x^{-2} - x^{-3} + 3x^{-5}$

$$f'(x) = -2x^{-3} + 3x^{-4} - 15x^{-6} \quad //$$

(5) $f(x) = \dfrac{4-x}{5-x^2}$

$$f'(x) = \frac{(4-x)'(5-x^2) - \overbrace{(4-x)(5-x^2)'}^{= +[+8x - 2x^2]}}{(5-x^2)^2}$$

$$f'(x) = \frac{-1(5-x^2) - (4-x)(-2x)}{(5-x^2)^2} = \frac{-1}{5-x^2} + \frac{8x - 2x^2}{(5-x^2)^2}$$

$$f'(x) = \frac{-x^2 + 8x - 5}{(5-x^2)^2} \quad //$$

⑧ $$f(x) = (x^3 + 2x - 1)^8$$

$$u(x) = x^3 + 2x - 1 \rightarrow f(u(x)) = (u(x))^8 = u^8$$

$$\frac{df(u)}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{du^8}{du}\frac{d}{dx}(x^3+2x-1)$$

$$\frac{df}{dx} = 8u^7(3x^2+2) = 8(3x^2+2)(x^3+2x-1)^7 \quad //$$

⑨ $$f(s) = 4\left(\frac{3s^2+2s}{2s+1}\right)^{-2} = 4\,\frac{(2s+1)^2}{(3s^2+2s)^2}$$

$$u(s) = \frac{3s^2+2s}{2s+1}, \quad f(u(s)) = 4u^{-2}$$

$$\frac{df}{ds} = \frac{df}{du}\frac{du}{ds} = \frac{d}{du}(4u^{-2})\,\frac{d}{ds}\left(\frac{3s^2+2s}{2s+1}\right)$$

$$\frac{df}{ds} = -8u^{-3}\left[\frac{(6s+2)(2s+1) - (3s^2+2s)\cdot 2}{(2s+1)^2}\right]$$

$$\frac{df}{ds} = -8u^{-3}\left[\frac{12s^2+10s+2-6s^2-4s}{(2s+1)^2}\right]$$

$$\frac{df}{ds} = -8\left(\frac{3s^2+2s}{2s+1}\right)^{-3}\left(\frac{6s^2+6s+2}{(2s+1)^2}\right)$$

$$\frac{df}{ds} = -8\,\frac{(2s+1)^{\cancel{3}}}{(3s^2+2s)^3}\cdot\frac{(6s^2+6s+2)}{\cancel{(2s+1)^2}}$$

$$\frac{df}{ds} = -8\,\frac{(2s+1)(6s^2+6s+2)}{(3s^2+2s)^3} \quad //$$

(11) $f(x) = \dfrac{2\sqrt{x^2+3}}{x+1} = \dfrac{2\,(x^2+3)^{1/2}}{x+1}$

$$f' = 2\left\{\left[\frac{d}{dx}(x^2+3)^{1/2}\right](x+1) - (x^2+3)^{1/2}\frac{d}{dx}(x+1)\right\} \Big/ (x+1)^2$$

$$f' = 2\left\{(x+1)\,\frac{du^{1/2}}{du}\,\frac{du}{dx} - (x^2+3)^{1/2}\right\} \Big/ (x+1)^2$$

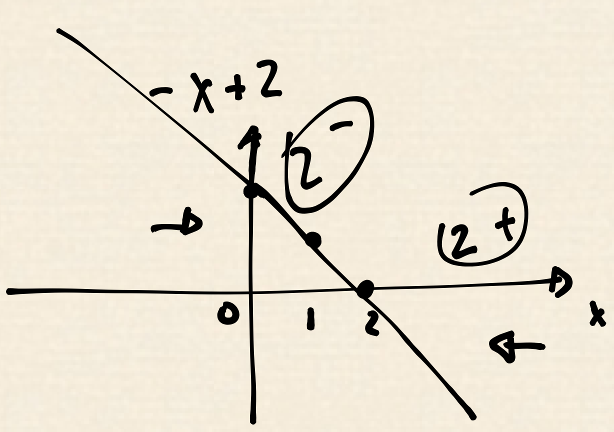
$$\boxed{\begin{aligned} u &= x^2+3 \\ \frac{du}{dx} &= 2x \end{aligned}}$$

$$f' = 2\left\{(x+1)\,\frac{1}{\cancel{2}}\,u^{-1/2}(\cancel{2}x) - (x^2+3)^{1/2}\right\} \Big/ (x+1)^2$$

$$f' = 2\left[\frac{(x^2+x)(x^2+3)^{-1/2} - (x^2+3)^{1/2}}{(x+1)^2}\right] //$$

(6) $f(t) = \dfrac{3}{t^4} + \dfrac{5}{t^5} = 3t^{-4} + 5t^{-5}$

$$\frac{df}{dt} = -12t^{-5} - 25t^{-6} = -\frac{12}{t^5} - \frac{25}{t^6} //$$

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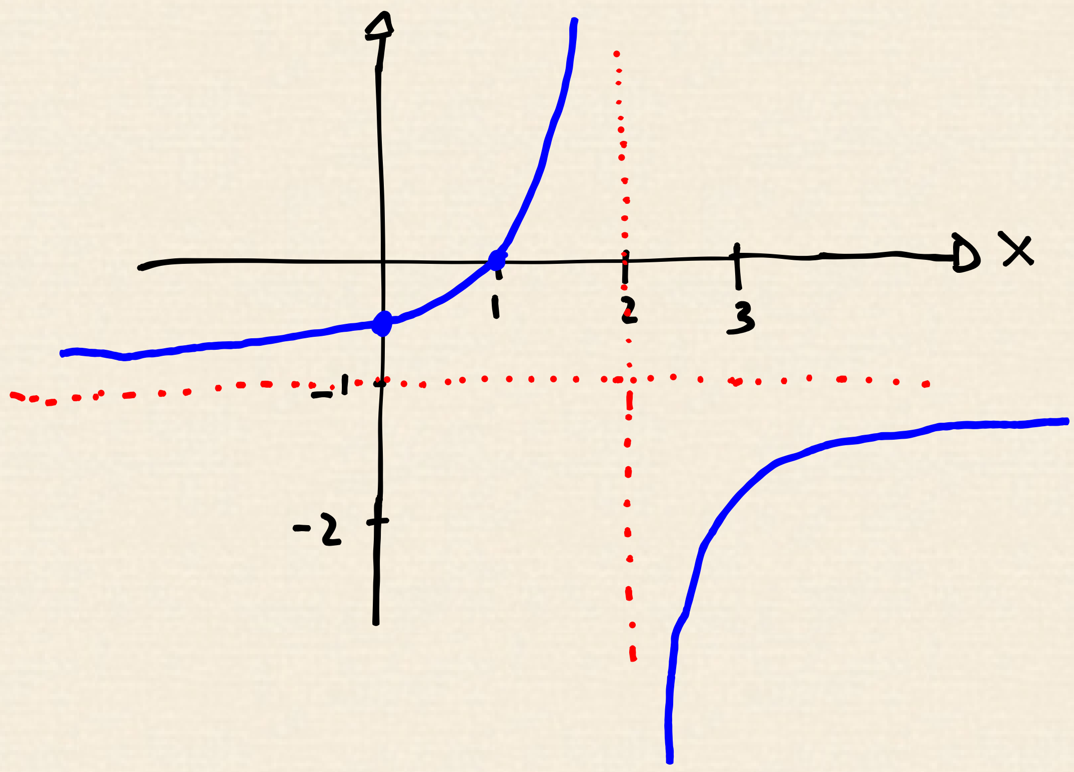
TESTS, questão (3) $f = \dfrac{x-1}{-x+2}$

A.H. $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \dfrac{\cancel{x}}{\cancel{x}} \cdot \dfrac{(1-\overset{\to 0}{1/x})}{(-1+\underset{\to 0}{2/x})} = -1$

A.V. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{x-1}{-x+2} = -\infty$ , $\lim_{x\to 2^-} f(x) = +\infty$

$f(0) = -1/2$

$f(x) = 0 \Rightarrow x = 1$



$f(x) = x^2 \Rightarrow f' = 2x$

$u = x \Rightarrow f(u) = u^2 \ , \ \frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \frac{du^2}{du}\frac{dx}{dx} = 2u = 2x$

$v = 2x+1$

$f(v) = (2x+1)^2 \ , \ \frac{df}{dx} = \frac{df}{dv}\frac{dv}{dx} = \frac{dv^2}{dv}\frac{d}{dx}(2x+1)$

$\Rightarrow \frac{df}{dx} = 2v\,2 = 4(2x+1) = 8x+4$

$f(v) = 4x^2 + 4x + 1$

$\frac{d}{dx} f(x) = 8x + 4$

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SEC. 7 (10)

$$f(t) = \frac{\overbrace{(x^2+x)^4}^{C(x)}}{(t^2+t+1)^7} = C(x)\,(t^2+t+1)^{-7}$$

$$\frac{df(t)}{dt} = C(x)\frac{d}{dt}(t^2+t+1)^{-7} = C(x)\frac{d}{dt}u(t)^{-7} = C(x)\frac{du^{-7}}{du}\frac{du}{dt}$$

$$= C(x)(-7)\,u^{-8}\,(2t+1)$$

$u = t^2+t+1$

$\frac{du}{dt} = 2t+1 \ \Rightarrow \ \frac{df}{dt} = -7\frac{(x^2+x)^4}{(t^2+t+1)^8}(2t+1)$ //

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Exemplo extra

$$f(t) = \frac{(t^2+t)^4}{(t^2+t+1)^7} = (t^2+t)^4\,(t^2+t+1)^{-7}$$

$$\frac{df}{dt} = 4(t^2+t)^3(2t+1)(t^2+t+1)^{-7} + (t^2+t)^4(-7)(t^2+t+1)^{-8}(2t+1)$$

$$= (t^2+t)^3\,(t^2+t+1)^{-7}\left[8t+4 \ -7(t^2+t)(t^2+t+1)(2t+1)\right]$$

$$\frac{d}{dx}\left[\frac{x^4}{(x^2+2)^1}\right] = \frac{d}{dx}\left[x^4\,(x^2+2)^{-1}\right] = 4x^3(x^2+2)^{-1} + x^4\,\frac{d}{dx}(x^2+2)^{-1}$$

$$= 4x^3(x^2+2)^{-1} + x^4\,(-1)\,(x^2+2)^{-2}\,2x$$

$$= \frac{4x^3}{x^2+2} - \frac{2x^5}{(x^2+2)^2}$$

$$= \frac{da^{-1}}{da}\,\frac{da}{dx}$$

$$= -1\,(x^2+2)^{-2}\,(2x)$$

Aula 8

## Derivada da Função Exponencial

$$f(x) = a^x \quad , \quad a > 0 \text{ e } a \neq 1.$$

$$\frac{d}{dx} a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a) \quad , \quad \boxed{\ln x = \log_e x}$$

$$\boxed{\frac{d}{dx} a^x = a^x \ln a}$$

Caso $a = e$ $\quad \frac{d}{dx} e^x = e^x \cancel{\ln e}^{\,1} \Rightarrow \frac{d}{dx} e^x = e^x$ //

## Derivada da Função Logarítmica

$$f(x) = \log_a x \quad , \quad a > 0 \text{ e } a \neq 1.$$

$$\frac{d}{dx} \log_a x = \lim_{h \to 0} \frac{\log_a(x+h) - \log_a(x)}{h} = \lim_{h \to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h}$$

$$u = \frac{h}{x} \Rightarrow \frac{d}{dx} \log_a x = \lim_{u \to 0} \frac{1}{x} \cdot \frac{1}{u} \log_a(1+u)$$

$h \to 0$, $u \to 0$

$$= \frac{1}{x} \lim_{u \to 0} \log_a (1+u)^{1/u} = \frac{1}{x} \lim_{v \to \infty} \log_a \left(1 + \frac{1}{v}\right)^v$$

$$\boxed{\lim_{v \to \pm\infty} \left(1 + \frac{1}{v}\right)^v = e} \quad v = \frac{1}{u} \, , \; u \to 0^+ \, , \; v \to +\infty$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x}\lim_{v\to+\infty}\log_a\left(1+\frac{1}{v}\right)^v$$

$$= \frac{1}{x}\log_a\left[\lim_{v\to\infty}\left(1+\frac{1}{v}\right)^v\right] = \frac{1}{x}\log_a e$$

$$\boxed{\frac{d}{dx}\log_a(x) = \frac{1}{x}\log_a e}$$

CASO $a=e \Rightarrow \frac{d}{dx}\ln(x) = \frac{1}{x}\cancelto{1}{\ln e} \Rightarrow \boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$ //

EXEMPLOS

Ex. 1 $f(x) = 2^x$ , $g(x) = \log_2 x$

$\frac{d}{dx}(2^x) = 2^x \ln 2$ // $\frac{d}{dx}\log_2 x = \frac{1}{x}\log_2 e$ //

Ex. 2 $\frac{d}{dx}e^x = e^x$ , $\frac{d}{dx}\ln x = \frac{1}{x}$

Ex. 3 $f(x) = 2^{2x^2+3x-1}$ , $u = 2x^2+3x-1$, $f = 2^u$

$$\frac{d}{dx}f(x) = \frac{d}{du}(2^u)\frac{du}{dx} = (4x+3)2^u\ln 2$$

$$\frac{d}{dx}f(x) = (4x+3)2^{2x^2+3x-1}\ln 2$$ //

Ex.4 $f(x) = \exp\left(\frac{x+1}{x-1}\right)$, $u = \frac{x+1}{x-1}$, $f = e^u$

$$\frac{d}{dx} f(x) = \frac{d\,e^u}{du}\frac{du}{dx} = e^u \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$\frac{d}{dx}\left(\frac{x+1}{x-1}\right) = \frac{1\cdot(x-1) - (x+1)\cdot 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$\frac{d}{dx} f(x) = \frac{-2}{(x-1)^2} \cdot \exp\left(\frac{x+1}{x-1}\right) \quad //$$

Ex.5 $f(x) = \log_2(3x^2 + 7x - 1)$, $u = 3x^2 + 7 - 1$, $f = \log_2 u$

$$\frac{d}{dx} f(x) = \frac{d}{du}\log_2 u \; \frac{d}{dx}(3x^2 + 7x - 1) = (6x + 7)\frac{1}{u}\log_2 e$$

$$\frac{d}{dx} f(x) = \frac{6x+7}{3x^2 + 7 - 1} \cdot \log_2 e \quad //$$

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$$\frac{d}{dx} e^{u(x)} = \frac{du}{dx} e^u$$

$$\frac{d}{dx} \ln(u(x)) = \frac{1}{u}\frac{du}{dx}$$

AULA 9

# Derivadas de Funções Trigonométricas

## FUNÇÃO SENO:

$$\frac{d}{dx} sen(x) = \lim_{h \to 0} \frac{sen(x+h) - sen(x)}{h}$$

$$sen(x+h) = sen(x)cos(h) + sen(h)cos(x)$$

$$sen(x+h) - sen(x) = sen(x)\left[cos(h) - 1\right] + sen(h)cos(x)$$

$$\Rightarrow \frac{d}{dx} sen(x) = \lim_{h \to 0} \left[ sen(x) \frac{cos(h)-1}{h} + cos(x) \frac{sen(h)}{h} \right]$$

$$= sen(x) \lim_{h \to 0} \frac{cos(h)-1}{h} + cos(x) \lim_{h \to 0} \cancel{\frac{sen(h)}{h}} \to 1$$

$$\lim_{h \to 0} \frac{(cos(h)-1)}{h} \frac{(cos(h)+1)}{(cos(h)+1)} = \lim_{h \to 0} \frac{cos^2(h) - 1}{h(cos(h)+1)} = \lim_{h \to 0} \frac{sen(h)}{h} \frac{sen(h)}{cos(h)+1}$$

$$\lim_{h \to 0} \frac{cos(h)-1}{h} = 0$$

$$\Rightarrow \boxed{\frac{d}{dx} sen(x) = cos(x)}$$

## FUNÇÃO COSSENO:

$$\boxed{\frac{d}{dx} cos(x) = -sen(x)}$$

## FUNÇÃO TANGENTE:

$$tg(x) = \frac{sen(x)}{cos(x)}$$

$$\frac{d}{dx} tg(x) = \left( \frac{d\, sen(x)}{dx} \cdot cos(x) - sen(x) \frac{d\, cos(x)}{dx} \right) / cos^2(x)$$

$$= \frac{cos^2(x) + sen^2(x)}{cos^2(x)} = \frac{1}{cos^2(x)} \qquad \boxed{\frac{d}{dx} tg(x) = sec^2(x)}$$

FUNÇÃO SECANTE: $\sec(x) = \dfrac{1}{\cos(x)} = (\cos(x))^{-1}$

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}(\cos(x))^{-1} = \frac{d\,u^{-1}}{du}\frac{du}{dx} = -u^{-2}\frac{d}{dx}\cos(x)$$

$$= -\frac{1}{\cos^2(x)}\cdot(-\text{sen}(x)) = \frac{\text{sen}(x)}{\cos(x)}\cdot\frac{1}{\cos(x)}$$

$$\boxed{\frac{d}{dx}\sec(x) = \text{tg}(x)\cdot\sec(x)}$$

FUNÇÃO COTANGENTE: $\text{cotg}(x) = \dfrac{1}{\text{tg}(x)} = \dfrac{\cos(x)}{\text{sen}(x)}$

$$\boxed{\frac{d}{dx}\text{cotg}(x) = -\text{cosec}^2(x)}$$

FUNÇÃO COSSECANTE: $\text{cosec}(x) = \dfrac{1}{\text{sen}(x)}$

$$\boxed{\frac{d}{dx}\text{cosec}(x) = -\text{cosec}(x)\cdot\text{cotg}(x)}$$

Ex.1 $f(x) = \text{sen}(x^3+x^2)$

$$\frac{d}{dx}f(x) = \frac{d}{du}\text{sen}(u)\frac{du}{dx}, \quad u = x^3+x^2$$

$$f'(x) = \cos(x^3+x^2)\cdot(3x^2+2x) //$$

Ex.2 $f(x) = \cos\left(\sqrt{x^3+2}\right)$

$$\frac{df}{dx} = \frac{d\cos(u)}{du}\frac{du}{dx}, \quad u = \sqrt{x^3+2}, \quad \frac{du}{dx} = \frac{d\sqrt{v}}{dv}\frac{dv}{dx}, \quad v = x^3+2$$

$$\Rightarrow f'(x) = -\text{sen}\left(\sqrt{x^3+2}\right)\cdot\frac{3}{2}\frac{x^2}{\sqrt{x^3+2}} // \qquad \frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{x^3+2}}\cdot 3x^2$$

Ex.3 $f(x) = \sec(x)\,\text{sen}(x^2)$

$$f'(x) = \frac{d\sec(x)}{dx} \cdot \text{sen}(x^2) + \sec(x)\,\frac{d\,\text{sen}(x^2)}{dx}$$

$$\frac{d}{dx}\sec(x) = \frac{d}{dx}\left(\cos(x)\right)^{-1} = -\cos^{-2}(x)\left(-\text{sen}(x)\right) = tg(x)\sec(x)$$

$$\frac{d}{dx}\text{sen}(x^2) = 2x\cos(x^2)$$

$$f'(x) = tg(x)\sec(x)\,\text{sen}(x^2) + 2x\cos(x^2)\sec(x)$$ //

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## Derivadas de Funções Hiperbólicas

SENO HIPERBÓLICO: $\text{senh}(x) = \dfrac{e^x - e^{-x}}{2}$

$$\frac{d}{dx}\text{senh}(x) = \frac{1}{2}\frac{d}{dx}\left(e^x - e^{-x}\right) = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$\boxed{\frac{d}{dx}\text{senh}(x) = \cosh(x)}$$

COSSENO HIPERBÓLICO: $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$$\frac{d}{dx}\cosh(x) = \frac{1}{2}\frac{d}{dx}\left(e^x + e^{-x}\right) = \frac{1}{2}\left(e^x - e^{-x}\right) = \text{senh}(x)$$

$$\boxed{\frac{d}{dx}\cosh(x) = \text{senh}(x)}$$

AULA 10

# Derivada de Função Inversa

$u = f(x)$ , $v = f^{-1}(x)$ . $v$ é a função inversa de $u$.

$\frac{d}{dx}$ $u(v(x)) = x$ , $f(f^{-1}(x)) = x$ .

$v(u(x)) = x$ , $f^{-1}(f(x)) = x$ .

$$\frac{d}{dx} u(v(x)) = \frac{du}{dv} \frac{dv}{dx} = \frac{dx}{dx} = 1$$

$$\boxed{\frac{du}{dv} \frac{dv}{dx} = 1}$$

Ex. $u = x^2$ , $D_u = [0, +\infty)$

$x^2 = u \rightarrow |x| = \sqrt{u} \rightarrow x = \sqrt{u} \rightarrow v(x) = \sqrt{x} = x^{1/2}$

$u(v(x)) = v^2(x) = (\sqrt{x})^2 = x$ $\qquad \frac{dv}{dx} = \frac{1}{2} \frac{1}{\sqrt{x}}$ //

$$\frac{du}{dx} = \frac{du}{dv} \frac{dv}{dx} = 1 \Rightarrow \frac{d v^2}{dv} \left( \frac{dv}{dx} \right) = 1$$

$$2v \, v' = 1 \rightarrow v' = \frac{1}{2v} = \frac{1}{2\sqrt{x}}$$

$$\frac{dv}{dx} = \frac{1}{2\sqrt{x}} \; //$$

DERIVADA DO ARCOSSENO: $sen^{-1}(x) = arcsen(x) = v(x)$

Para $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $sen(sen^{-1}(x)) = x$, $\boxed{sen(v) = x}$

$u = sen(x)$ $\qquad u(v(x)) = x$

$$\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dx} = 1, \quad \frac{du}{dv} = \frac{d}{dv}u(v) = \frac{d}{dv}sen(v) = cos(v)$$

$$\Rightarrow cos(v)\, v' = 1 \rightarrow v' = \frac{1}{cos(v)}, \quad cos(v) = \sqrt{1 - \underline{sen^2(v)}}$$

$$cos(v) = \sqrt{1 - x^2}$$

$$\boxed{\frac{d}{dx} sen^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}}$$

Ver derivada de $arccos(x)$, $arctg(x)$, $arcsec(x)$, $arccosec(x)$, $arccotg(x)$.

## Derivadas de Ordem Superior

Derivada Primeira $\quad f'(x) = \frac{d f(x)}{dx}$

Derivada Segunda $\quad f''(x) = \frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right)$

Ex. $\quad x(t) = \frac{x_0}{2} + v_0(t - t_0) + \frac{x_0}{2} exp\left(-\frac{t - t_0}{t_0}\right)$, $t > t_0$

Qual é a força? $\quad F = m a = m \frac{dv}{dt} = m \frac{d}{dt}\left(\frac{dx(t)}{dt}\right)$

$$v(t) = \frac{d}{dt}x(t) = v_0 + \frac{x_0}{2}\left(-\frac{1}{t_0}\right) exp\left(-\frac{t - t_0}{t_0}\right)$$

$$a = \frac{d^2}{dt^2}x(t) = \frac{d}{dt}\left(\frac{dx(t)}{dt}\right) = \frac{d}{dt}v(t) = \frac{x_0}{2}\frac{1}{t_0^2} exp\left(-\frac{t - t_0}{t_0}\right)$$

$$F = \frac{m\, x_0}{2 t_0^2} exp\left(-\frac{t - t_0}{t_0}\right)$$

# Derivada de Função Implícita

$$F(x,y) = 0 \quad , \; y = f(x) \text{ - definido implicitamente.}$$

<u>Ex.1</u> $F(x,y) = x^2 + \frac{y}{2} - 1 = 0 \quad , \; y' ?$

$$\frac{d}{dx} F(x, y(x)) = 0 \Rightarrow \frac{d}{dx}\left[x^2 + \frac{y}{2} - 1\right] = \frac{d}{dx} 0$$

$$2x + \frac{y'}{2} = 0 \Rightarrow \boxed{y' = -4x}$$

Explicitamete: $F(x,y)=0 \Rightarrow \frac{y}{2} = 1 - x^2 \Rightarrow y = 2 - 2x^2$

$$y' = -4x$$

<u>Ex.2</u> $F(x,y) = y^2 + x^2 - 4 = 0 \quad , \; y \geq 0 \quad , \; y' ?$

$$\frac{d}{dx}\left[F(x,y) = 0\right] \Rightarrow \frac{d}{dx} y^2 + \frac{d}{dx} x^2 = 0$$

$$\frac{dy^2}{dy} \frac{dy}{dx} + 2x = 0 \Rightarrow \cancel{2} y y' + \cancel{2} x = 0$$

$$\Rightarrow \boxed{y' = \frac{-x}{y}}$$

Explicitamente: $y = \sqrt{4 - x^2}$

$$y' = \frac{1}{\cancel{2}} (4 - x^2)^{-1/2} (-\cancel{2}x)$$

$$y' = -\frac{x}{\sqrt{4 - x^2}} = -\frac{x}{y}$$

Exercícios da lista, SEÇÃO 7.

(12) $f(x) = 5^{\sqrt{x^2+1}}$ ,

$$\boxed{\begin{array}{l} \dfrac{d}{dx} a^x = a^x \ln(a) \\ \text{Se } a = e : \quad \dfrac{d}{dx} e^x = e^x \end{array}}$$

$$\frac{d}{dx} f(x) = \frac{d(5^u)}{du} \frac{du}{dx}$$

$$= 5^u \ln 5 \, \frac{d}{dx}\sqrt{x^2+1} = \ln(5)\, 5^{\sqrt{x^2+1}} \, \frac{1}{\cancel{2}} (x^2+1)^{-1/2} \cancel{2}x$$

$$f'(x) = \ln(5) \frac{x\, 5^{\sqrt{x^2+1}}}{\sqrt{x^2+1}} \quad //$$

(16) $f(x) = \log_{10}(\sqrt{x^2+1})$ ,

$$\boxed{\begin{array}{l} \dfrac{d}{dx} \log_a(x) = \dfrac{1}{x} \log_a e \\ \text{Se } a = e : \quad \dfrac{d}{dx} \ln(x) = \dfrac{1}{x} \end{array}}$$

$$\frac{d}{dx} f(x) = \frac{d}{du} \log_{10}(u) \frac{du}{dx}$$

$$= \frac{1}{u} \log_{10} e \cdot \frac{d}{dx}\sqrt{x^2+1} = \frac{\log_{10} e}{\sqrt{x^2+1}} \frac{x}{\sqrt{x^2+1}}$$

$$f'(x) = \frac{\log_{10} e \; x}{x^2+1} \quad //$$

(17) $f(x) = e^{x+1} \cdot \log_{10}\left(\frac{x+1}{x}\right)$

$$\frac{x+1}{x} = 1 + x^{-1}$$

$$f'(x) = \frac{d}{dx} e^{x+1} \cdot \log_{10}\left(\frac{x+1}{x}\right) + e^{x+1} \frac{d}{dx} \log_{10}\left(\frac{x+1}{x}\right)$$

$$\frac{d}{dx} x^{-1} = -1 x^{-2}$$

$$f'(x) = 1 \cdot e^{x+1} \log_{10}\left(\frac{x+1}{x}\right) + e^{x+1} \frac{d}{du} \log_{10}(u) \cdot \frac{d}{dx}\left(1 + x^{-1}\right)$$

$$f'(x) = e^{x+1}\left[\log_{10}\left(\frac{x+1}{x}\right) + \frac{\log_{10} e}{1+x^{-1}} \left(-x^{-2}\right)\right]$$

$$\frac{x^{-2}}{1+x^{-1}} = \frac{1}{x^2} \cdot \frac{1}{(1+x^{-1})}$$

$$f'(x) = e^{x+1}\left[\log_{10}\left(1+x^{-1}\right) - \log_{10} e \cdot \frac{1}{x^2+x}\right] \quad //$$

(19) $f(x) = \dfrac{\ln(x^4+2x^2)}{(4x^3+4x)}$ ,

$$\frac{df}{dx} = \frac{\frac{1}{x^4+2x^2}(4x^3+4x)(4x^3+4x) - \ln(x^4+2x^2)(12x^2+4)}{(4x^3+4x)^2}$$

$$\frac{df}{dx} = \frac{1}{x^4+2x^2} - \frac{(12x^2+4)}{(4x^3+4x)^2}\ln(x^4+2x^2)$$

$$\frac{12x^2+4}{(4x^3+4x)^2} = \frac{4(3x^2+1)}{4^2(x^3+x)^2} = \frac{1}{4}\frac{(3x^2+1)}{(x^3+x)^2}$$

(22) $f(x) = \cos\left(\frac{x^4}{4}\right)$

$$f'(x) = \frac{d}{du}\cos(u)\ \frac{d}{dx}\left(\frac{x^4}{4}\right) = -\operatorname{sen}(u)\ x^3$$

$$f'(x) = -\,x^3 \operatorname{sen}\left(\frac{x^4}{4}\right) //$$

(23) $f(x) = \cos(\operatorname{sen}(x^2))$ $\quad u = \operatorname{sen}(x^2)\ ,\quad v = x^2$

$$f'(x) = \frac{d}{du}\cos(u) \cdot \frac{d}{dx}\operatorname{sen}(x^2) = -\operatorname{sen}(u) \cdot \frac{d}{dv}\operatorname{sen}(v) \cdot \frac{dx^2}{dx}$$

$$f'(x) = -\operatorname{sen}(\operatorname{sen}(x^2))\cos(x^2)(2x) //$$

$$\boxed{\frac{d}{dx}e^{-x} = \frac{d}{du}e^{u}\,\frac{d(-x)}{dx} = -e^{-x}}$$

(24) $f(x) = \operatorname{sen}(e^{-x} \cdot x^2)$

$$f'(x) = \frac{d}{du}\operatorname{sen}(u) \cdot \frac{d}{dx}(e^{-x} \cdot x^2) = \cos(e^{-x}x^2)\left[-e^{-x}x^2 + e^{-x}(2x)\right]$$

$$f'(x) = e^{-x}\cos(e^{-x}x^2)(-x^2 + 2x) //$$

(26) $f(x) = \operatorname{cosec}\left(\frac{x+1}{x-1}\right)$

$$f'(x) = \frac{d}{du}\operatorname{cosec}(u)\ \frac{d}{dx}\left(\frac{x+1}{x-1}\right)$$

$$\frac{d}{dx}\operatorname{cosec}(x) = \frac{d}{dx}(\operatorname{sen}(x))^{-1}$$

$$= -1(\operatorname{sen} x)^{-2}\cos x$$

$$= -\operatorname{cotg}(x) \cdot \operatorname{cosec}(x).$$

$$f'(x) = -\operatorname{cotg}\left(\frac{x+1}{x-1}\right)\operatorname{cosec}\left(\frac{x+1}{x-1}\right)\left[\frac{1\cdot(\cancel{x}-1) - (\cancel{x}+1)\cdot 1}{(x-1)^2}\right]$$

$$f'(x) = -\operatorname{cotg}\left(\frac{x+1}{x-1}\right)\operatorname{cosec}\left(\frac{x+1}{x-1}\right)\frac{(-2)}{(x-1)^2} //$$

(29) $f(x) = \operatorname{senh}(x^2 + x^4)$

$$f'(x) = \frac{d}{du}\operatorname{senh}(u)\,\frac{d}{dx}(x^2+x^4) = \cosh(x^2+x^4)\,(2x+4x^3)$$

$$f'(x) = (2x+4x^3)\cosh(x^2+x^4) //$$

(32) $f(x) = \operatorname{arctg}\left(\frac{x^2}{2+x}\right)$ . $\boxed{\frac{d}{dx}\operatorname{arctg}(x) = \frac{1}{1+x^2}}$

$$f'(x) = \frac{d}{du}\operatorname{arctg}(u) \cdot \frac{d}{dx}\left(\frac{x^2}{2+x}\right) = \frac{1}{1+u^2}\left[\frac{\overbrace{2x(2+x) - x^2\cdot 1}^{4x+2x^2}}{(2+x)^2}\right]$$

$$f'(x) = \frac{x^2+4x}{(2+x)^2} \cdot \frac{1}{1+\frac{x^4}{(2+x)^2}} = \frac{x^2+4x}{\cancel{(2+x)^2}} \cdot \frac{1}{\frac{(2+x)^2+x^4}{\cancel{(2+x)^2}}}$$

$$f'(x) = \frac{x^2+4x}{(2+x)^2+x^4} //$$

SEC. 8, (2) $f(t) = x_0 \operatorname{sen}(\omega t + a)$

$$\frac{df}{dt} = x_0\,\omega \cos(\omega t + a)$$

$$\frac{d^2}{dt^2}f(t) = \frac{d}{dt}\left(\frac{df}{dt}\right) = \frac{d}{dt}\left(x_0\omega\cos(\omega t + a)\right) = -x_0\omega^2\operatorname{sen}(\omega t + a) //$$

③ $f(t) = x_0 e^{-wt} \cos(wt+a)$

$$\frac{df}{dt} = x_0 \left[ -w e^{-wt} \cos(wt+a) + e^{-wt}(-w) \operatorname{sen}(wt+a) \right]$$

$$\frac{df}{dt} = -x_0 w e^{-wt} \left[ \cos(wt+a) + \operatorname{sen}(wt+a) \right]$$

$$\frac{d^2}{dt^2} f(t) = -x_0 w \left\{ -w e^{-wt} \left[ \cos(wt) + \operatorname{sen}(wt+a) \right] + e^{-wt} \left[ -w \operatorname{sen}(wt+a) + w \cos(wt+a) \right] \right\}$$

$$\frac{d^2}{dt^2} f(t) = -x_0 w^2 e^{-wt} \left[ \cancel{-\cos(wt+a)} - \operatorname{sen}(wt+a) - \operatorname{sen}(wt+a) + \cancel{\cos(wt+a)} \right]$$

$$\frac{d^2}{dt^2} f(t) = 2 x_0 w^2 e^{-wt} \operatorname{sen}(wt+a) \quad //$$

$F = ma$

$-kx = m \frac{d^2}{dt} x(t)$

$$\boxed{\ddot{x} + \frac{k}{m} x = 0}$$

SEC. 9

② $\frac{d}{dx}\left[ x y^2 + 2y^3 = x - 2y \right]$

$\frac{dy^2}{dx} = \frac{dy^2}{dy} \frac{dy}{dx} = 2yy'$

$\frac{dy^3}{dx} = 3y^2 y'$

$$\Rightarrow 1 \cdot y^2 + x \frac{dy^2}{dx} + 2 \frac{dy^3}{dx} = 1 - 2y'$$

$$\Rightarrow y^2 + 2xyy' + 6y^2 y' = 1 - 2y'$$

$$\Rightarrow y'(2xy + 6y^2 + 2) = 1 - y^2$$

$$y' = \frac{1-y^2}{2xy+6y^2+2} \quad //$$