proj$_u$ v

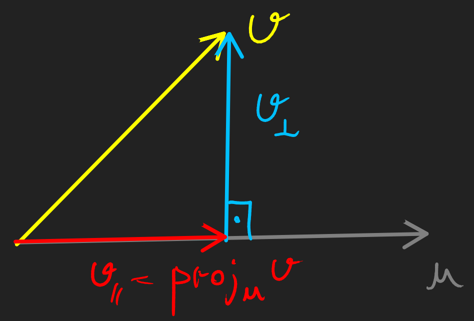
$proj_u v$

$\log_{10} x$

Objetivo:

9. No espaço $\mathbb{R}^3$ com produto interno usual, utilize o processo de ortonormalização de Gram-Schmidt para obter uma base ortonormal partindo dos vetores $\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (1, 2, 1)$ e $\boldsymbol{u}_3 = (0, 0, 1)$.

Projeção




$$v = v_\parallel + v_\perp \Rightarrow v_\perp = v - v_\parallel$$

$$v_\parallel = \alpha u = proj_u v$$

$$\langle v_\perp, u \rangle = 0$$

$$\langle v_\perp, u \rangle = 0$$

$$\langle v - v_\parallel, u \rangle = 0$$

$$\langle v - \alpha u, u \rangle = 0$$

$$\langle v, u \rangle - \alpha \langle u, u \rangle = 0$$

$$\langle v, u \rangle = \alpha \langle u, u \rangle$$

$$\frac{\langle v, u \rangle}{\langle u, u \rangle} = \alpha \rightarrow$$ COEFICIENTE DE FOURIER de v sobre u

$$proj_u v = \frac{\langle v, u \rangle}{\langle u, u \rangle} u$$

$$v_\perp = v - \frac{\langle v, u \rangle}{\langle u, u \rangle} u$$

Base ortonormal

$$B = \{v_1, v_2, \ldots, v_n\}$$

$$\langle v_i, v_j \rangle = 0 \quad se\ i \neq j$$ Ortogonal

$$\| v_i \| = 0$$ normal (normalizada)

$$\langle v_i, v_i \rangle = 0$$

$$\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 1 & se\ i = j \\ 0 & se\ i \neq j \end{cases}$$ ortonormal

Seja $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

$B = \{v_1, \dots, v_n\}$ ortonormal

$$[v]_B = \begin{bmatrix}\alpha_1\\ \alpha_2 \\ \vdots \\ \alpha_n\end{bmatrix}$$

$$\langle v, v_1\rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n, v_1\rangle$$
$$= \alpha_1 \underbrace{\langle v_1, v_1\rangle}_{1} + \alpha_2 \underbrace{\langle v_2, v_1\rangle}_{0} + \dots + \alpha_n \underbrace{\langle v_n, v_1\rangle}_{0}$$
$$= \alpha_1 + 0 + \dots + 0$$

$\alpha_1 = \langle v, v_1\rangle$

$\alpha_2 = \langle v, v_2\rangle$

$\vdots$

$\|v\| = \sqrt{\alpha_1^2 + \alpha_2^2 + \dots \alpha_n^2}$

$w = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

$\langle v, w\rangle = \alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n$

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Teorema: Todo espaço vetorial de dimensão finita possui bases ortonormal.

Prova: é possível efeituar o processo de Gram-Schmidt

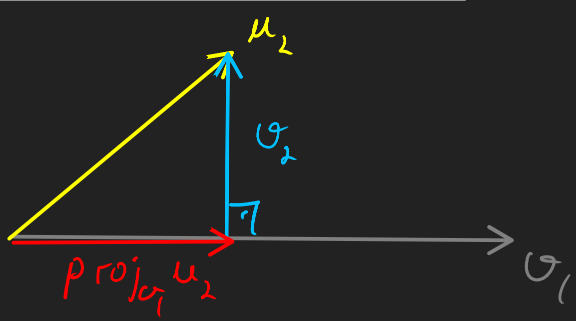
Ex.:

9. No espaço $\mathbb{R}^3$ com produto interno usual, utilize o processo de ortonormalização de Gram-Schmidt para obter uma base ortonormal partindo dos vetores $\boldsymbol{u}_1 = (1, 1, 1)$, $\boldsymbol{u}_2 = (1, 2, 1)$ e $\boldsymbol{u}_3 = (0, 0, 1)$.

$B = \{u_1, u_2, u_3\}$

$C = \{v_1, v_2, v_3\}$ ortogonal

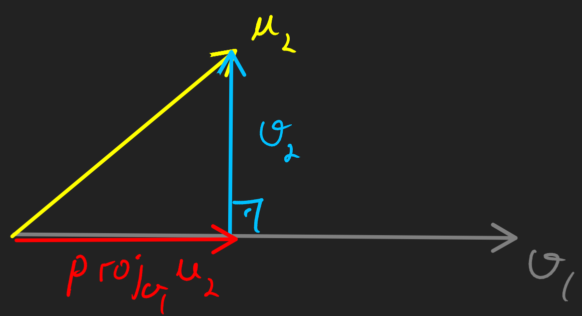
$G = \left\{\frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|}\right\}$




$C = \left\{(1,1,1), \frac{1}{3}(-1,2,-1), \frac{1}{2}(-1,0,1)\right\}$

$v_1 = u_1 = (1,1,1)$

$$v_2 = u_2 - \mathrm{proj}_{v_1} u_2$$



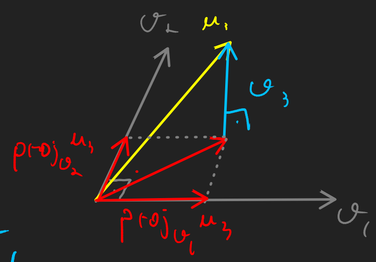

$$= u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= (1,2,1) - \frac{\langle (1,2,1), (1,1,1) \rangle}{3} (1,1,1)$$

$$= (1,2,1) - \frac{1+2+1}{3} (1,1,1) = (1,2,1) - \frac{4}{3} (1,1,1)$$

$$v_2 = \left( \frac{-1}{3}, \frac{2}{3}, \frac{-1}{3} \right) = \frac{1}{3} (-1, 2, -1)$$

$$v_3 = u_3 - \mathrm{proj}_{v_2} u_3 - \mathrm{proj}_{v_1} u_3$$




$$= u_3 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$\langle u_3, v_2 \rangle = \langle (0,0,1), \tfrac{1}{3}(-1,2,-1) \rangle = \frac{1}{3}(-1) = -\frac{1}{3}$$

$$\langle v_2, v_2 \rangle = \frac{1}{3} \frac{1}{3} \left( (-1)^2 + 2^2 + (-1)^2 \right) = \frac{1}{9} (1+4+1) = \frac{2}{3}$$

$$\langle u_3, v_1 \rangle = \langle (0,0,1), (1,1,1) \rangle = 1$$

$$\langle v_1, v_1 \rangle = \langle (1,1,1), (1,1,1) \rangle = 3$$

$$= (0,0,1) - \left( \frac{-1/3}{2/3} \right) \frac{1}{3} (-1,2,-1) - \frac{1}{3} (1,1,1)$$

$$= (0,0,1) - \frac{-1}{6} (-1,2,-1) - \frac{1}{3} (1,1,1)$$

$$= \left( -\frac{1}{6} - \frac{1}{3}, -\frac{-2}{6} - \frac{1}{3}, 1 - \frac{1}{6} - \frac{1}{3} \right) = \left( \frac{-1}{2}, 0, \frac{1}{2} \right)$$

$$C = \left\{ (1,1,1),\ \frac{1}{3}(-1,2,-1),\ \frac{1}{2}(-1,0,1) \right\}$$

é base ortogonal de $\mathbb{R}^3$

Normalizar:

$$G = \left\{ \frac{(1,1,1)}{\|(1,1,1)\|},\ \frac{\cancel{\frac{1}{3}}(-1,2,-1)}{\|\cancel{\frac{1}{3}}(-1,2,-1)\|},\ \frac{\frac{1}{2}(-1,0,1)}{\|\frac{1}{2}(-1,0,1)\|} \right\}$$

$$G = \left\{ \frac{1}{\sqrt{3}}(1,1,1),\ \frac{1}{3}\,\frac{(-1,2,-1)}{\sqrt{2}/\sqrt{3}},\ \frac{1}{\sqrt{2}}(-1,0,1) \right\}$$

$$\left\|\frac{1}{3}(-1,2,-1)\right\| = \frac{1}{3}\|(-1,2,-1)\| = \frac{1}{3}\sqrt{1+4+1} = \frac{1}{3}\sqrt{6} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$G = \left\{ \frac{1}{\sqrt{3}}(1,1,1),\ \frac{1}{\sqrt{6}}(-1,2,-1),\ \frac{1}{\sqrt{2}}(-1,0,1) \right\}$$

G é base ORTONORMAL de $\mathbb{R}^3$.

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$M_{2\times 2}$ Prod. interno: $\langle u, v\rangle = tr(v^T u)$

$$B = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \right\}$$

$$u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} \quad v = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \quad tr(v^T u) = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$M_{2\times 2}$ Prod. interno: $\boxed{\langle u, v\rangle = tr(v^T u)}$

$$B = \left\{ \underset{v_1}{\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 2 & 1 \\ -4 & 0 \end{pmatrix}}, \underset{v_3}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}, \underset{v_4}{\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}} \right\}$$

a) Mostre que B é ortogonal, mas não é ortonormal

b) Escreva as coordenadas da seguinte matriz na base B

$$\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix}$$

a) $0 = \langle v_1, v_2 \rangle = tr\left( \begin{pmatrix} 2 & 1 \\ -4 & 0 \end{pmatrix}^T \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right) = 2(1) + 1(2) + (-4)(1) + 0 \cdot 0 = 2 + 2 - 4 = 0$ //

$0 = \langle v_1, v_3 \rangle$

$0 = \langle v_1, v_4 \rangle$

$0 = \langle v_2, v_3 \rangle$ (Exercício)

$0 = \langle v_2, v_4 \rangle$

$0 = \langle v_3, v_4 \rangle$

Com isso, B é ortogonal.

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = \left\langle \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right\rangle = 1^2 + 2^2 + 1^2 + 0^2 = 6 \neq 1$$

Logo B não é normalizada.

$$\left[ \underset{v}{\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix}} \right]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

$$\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 & 1 \\ -4 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_4 = 5 \\ 2\alpha_1 + \alpha_2 - 2\alpha_4 = 3 \\ \alpha_1 - 4\alpha_2 + \alpha_4 = -2 \\ \alpha_3 = 4 \end{cases}$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$$

$$\boxed{\langle v, v_1 \rangle} = \alpha_1 \langle v_1, v_1 \rangle$$

$$\alpha_1 = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} = \frac{\left\langle \begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \right\rangle}{6}$$

$$\alpha_1 = \frac{5 + 6 - 2 + 0}{6} = \frac{9}{6} = \frac{3}{2}$$

$$\alpha_2 = \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle} = 1$$

$$\alpha_3 = \frac{\langle v, v_3 \rangle}{\langle v_3, v_3 \rangle} = 4$$

$$\alpha_4 = \frac{\langle v, v_4 \rangle}{\langle v_4, v_4 \rangle} = \frac{7}{14} = \frac{1}{2}$$

$$\left[ \begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix} \right]_B = \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 1/2 \end{bmatrix}.$$

3. Sejam $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$, com $\boldsymbol{u} = (u_1, u_2)$ e $\boldsymbol{v} = (v_1, v_2)$. Considerando o produto interno

$$\boxed{\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + 2u_2 v_2 - u_1 v_2 - u_2 v_1,}$$

(a) Determinar $m$ a fim de que os vetores $(1+m, 2)$ e $(3, m-1)$ sejam ortogonais.

(b) Determinar todos os vetores de $\mathbb{R}^2$ ortogonais a $(2, 1)$.

(c) Determinar todos os vetores da forma $(m, m-1)$ de norma igual a 1.

$$\langle (\overset{u_1}{1+m}, \overset{u_2}{2}), (\overset{v_1}{3}, \overset{v_2}{m-1}) \rangle = 0$$

$$\langle u, v \rangle$$

---

$P_2(\mathbb{R})$

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t) \, dt$$

$$B = \{1, t, t^2\}$$

$$u_1 = 1$$

$$u_2 = t - \text{proj}_1 t = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} 1$$

$$= t - \frac{\cancel{\int_{-1}^{1} t \, dt} \rightarrow 0}{\int_{-1}^{1} 1^2 \, dt} 1 = t$$

$$u_3 = t^2 - \text{proj}_t t^2 - \text{proj}_1 t^2$$

$$= t^2 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 \leftarrow$$

$$= t^2 - \frac{\cancel{\int_{-1}^{1} t^3\,dt}}{\int_{-1}^{1} t^2\,dt}\, t - \frac{\int_{-1}^{1} t^2\,dt}{\int_{-1}^{1} 1^2\,dt}\, 1$$

$0$

$$= t^2 - \frac{2/3}{2}\, 1 = t^2 - 1/3$$

$$C = \left\{ 1,\ t,\ t^2 - 1/3 \right\} \text{ ortogonal}$$

$$\left\{ \frac{1}{\sqrt{2}},\ \frac{\sqrt{3}}{\sqrt{2}}\, t,\ \frac{t^2 - 1/3}{\| t^2 - 1/3 \|} \right\} \qquad \| v \| \neq | v |$$

$$\hookrightarrow \sqrt{\langle t^2 - 1/3,\ t^2 - 1/3 \rangle}$$