Notas de aula ECT2202 T02 2022-02-08 Aula 23 — Diagonalização de f. quadraticas

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) =$

máximos/mínimos

$$\boxed{f_{xx} f_{xy} - f_{xy} \lessgtr 0}$$

Seja $g : \mathbb{R} \to \mathbb{R}$

Fórmula de Taylor para g

$$g(x) \approx g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2}(x - x_0)^2 + \cancel{O(x - x_0)^3}$$

big O notation

Se x_0 é ponto crítico:

$$g(x) \approx g(x_0) + \frac{g''(x_0)}{2}(x - x_0)^2$$

parábola com vértice em x_0 (ou reta horizontal)

Em duas variáveis: $f : \mathbb{R}^2 \to \mathbb{R}$

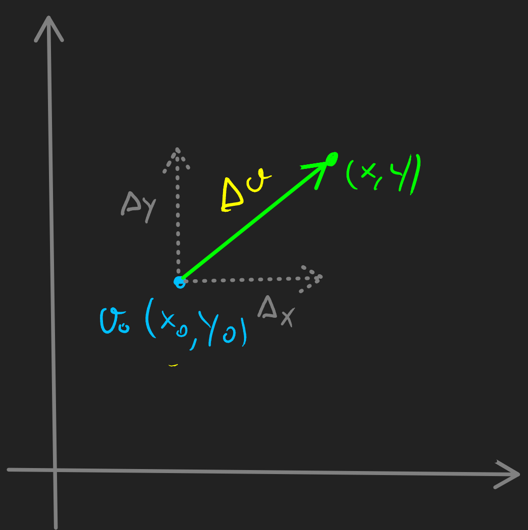
$\Delta x = x - x_0$

$\Delta y = y - y_0$

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \Delta y +$$

$$+ \frac{1}{2} \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} \Delta x^2 + \frac{1}{2} \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} \Delta y^2 +$$

$$+ \frac{1}{2}\, 2 \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x_0,y_0)} \Delta x \Delta y + O(\Delta x^3, \Delta y^3)$$



$$f(v) = f(v_0) + \langle \vec{\nabla} f(v_0), \Delta v \rangle + \frac{1}{2} \langle \Delta v \,|\, H \Delta v \rangle$$

$$[v_0] = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \qquad [\Delta v] = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$f(v) = f(v_0) + \begin{bmatrix} f_x & f_y \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} H \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

iguais

matriz hessiana de f em (x_0,y_0)

Se v_0 for ponto crítico $\vec{\nabla} f(v_0) = 0$

$$f(v) \approx f(v_0) + \underbrace{\frac{1}{2} \begin{bmatrix} \Delta x & \Delta y \end{bmatrix} H \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}}_{\text{forma quadrática}}$$

$H = [q]$ Existe base B de $\mathbb{R}^2$ em que $[q]$ é diagonal.

$$[\Delta v]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$f(v) \approx f(v_0) + \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

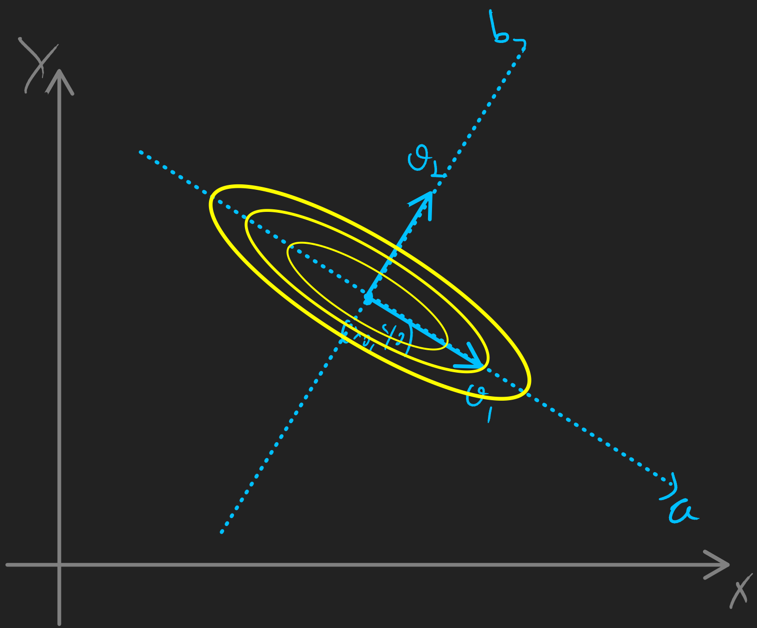
$$f(v) = f(v_0) + \lambda_1 a^2 + \lambda_2 b^2$$

Curvas de nível:

$$\boxed{\lambda_1 a^2 + \lambda_2 b^2 = k}$$

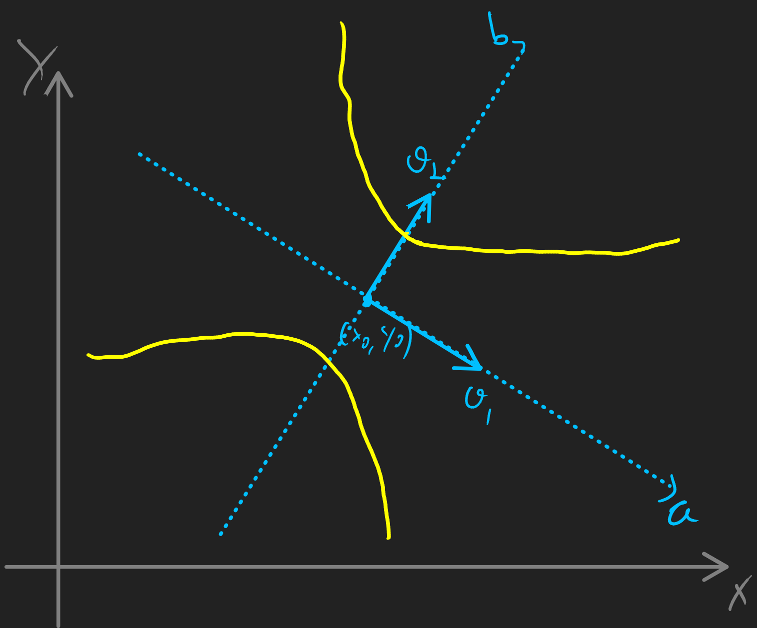
$$\frac{a^2}{(1/\lambda_1)} + \frac{b^2}{(1/\lambda_2)} = k$$

Elipse, hipérbole, parábola...



$B = \{v_1, v_2\}$

λ_1, λ_2 mesmo sinal



$$\frac{a^2}{(1/\lambda_1)} + \frac{b^2}{(1/\lambda_2)} = 1$$

$\lambda_2 < 0$

$\lambda_1 > 0$

$\lambda_2 < 0$

Ponto de sela

$$f(v) = f(v_0) + \lambda_1 a^2 + \lambda_2 b^2$$

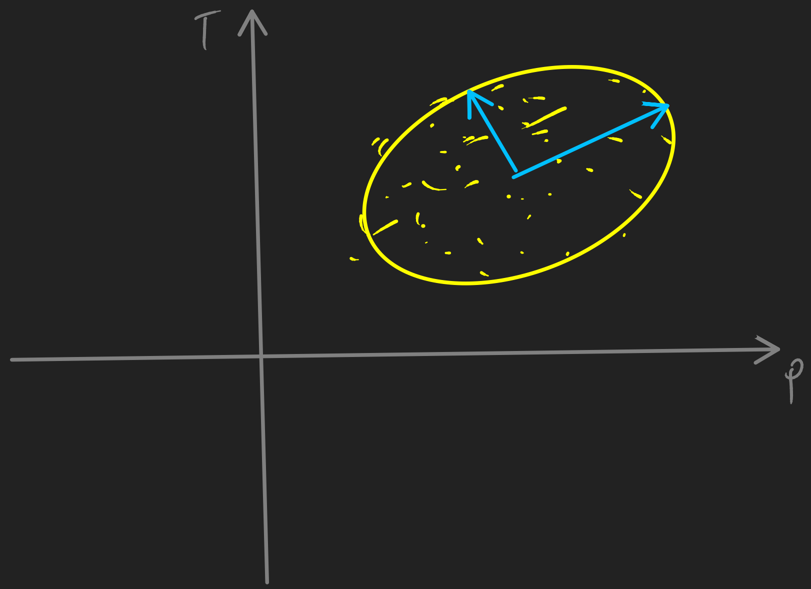
b = 0 deslocamento na direção a f(v) cresce mais rapidamente

a = 0 deslocamento na direção b: f(v) decresce mais rapidamente

Decomposição em valores singulares (SVD: singular value decomposition)
Análise de componente principal: (PCA: PRINCPIAL COMPONENT ANALYSIS)

(P_1, T_1)

(P_2, T_2)



$$f(x_1, x_2, x_3, \dots, x_n) \cong f(0) + [x_1 \cdots x_n][H]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

diagonalizar e

ordenar os autovalores em ordem decrescente de módulo

$$|\lambda_1| > |\lambda_2| > |\lambda_3| > \dots > |\lambda_n|$$

$$f(a_1, a_2, \dots, a_n) \cong f(0) + \lambda_1 a_1^2 + \lambda_2 a_2^2 + \dots + \lambda_n a_n^2$$

JPEG 8

MPEG

Exemplo: escreva a forma quadrática a seguir como uma soma de quadrados

$$q(a,b) = \pm a^2 \pm b^2$$

$$q(x,y) = x^2 - 10xy + y^2$$

$$q(x,y) = Ax^2 + Bxy + Cy^2$$

$$\langle (x,y),(a,b) \rangle$$

$$\downarrow$$

$$[q] = \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \leftarrow$$

matriz simétrica
numa base ortonormal:
será diagonal em alguma
base ortonormal

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = ax + by$$

$$\downarrow$$

fórmula de parseval

$$[q] = \begin{bmatrix} 1 & -10/2 \\ -10/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

Diagonalizar [q]

$$0 = p_q(\lambda) = \det \begin{bmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{bmatrix} =$$

$$0 = (1-\lambda)^2 - 25$$

$$0 = \lambda^2 - 2\lambda + 1 - 25$$

$$0 = \lambda^2 - 2\lambda - 24 \Rightarrow \begin{cases} \lambda_1 \lambda_2 = -24 \\ \lambda_1 + \lambda_2 = 2 \end{cases}$$

$$\lambda_1 = 6 \qquad \lambda_2 = -4$$

Autovetores de λ=6:

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x - 5y = 6x \\ -5x + y = 6y \end{cases} \Rightarrow \begin{cases} -5x - 5y = 0 \\ -5x - 5y = 0 \end{cases} \Rightarrow y = -x$$

$$v_1 = (x, -x) = x(1,-1)$$

quero base de autovetores ORTONORMAL

Escolho $v_1 = \frac{1}{\sqrt{2}}(1,-1)$ //

Autovetor com λ=-4

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -4\begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} x - 5y = -4x \\ -5x + y = -4y \end{cases}$$

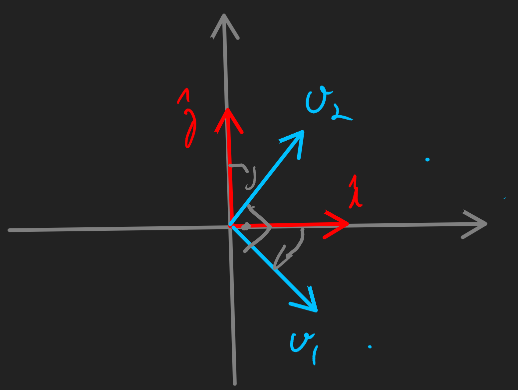
$$\begin{cases} 5x - 5y = 0 \\ -5x + 5y = 0 \end{cases} \Rightarrow x = y$$

$$v_2 = (x, x) = x(1,1)$$

escolho normalizar:

$$v_2 = \frac{1}{\sqrt{2}}(1,1)$$

$$B = \left\{ v_1 = \frac{1}{\sqrt{2}}(1,-1),\ v_2 = \frac{1}{\sqrt{2}}(1,1) \right\}$$



A base B é a base canônica rotacionada de -π/4

$$[q]_B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

$$q(v) = [v]_B^T [q]_B [v]_B = [a \ \ b] \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$q(v) = 6a^2 - 4b^2.$$

↳ Forma quadrática diagonalizada.

Ainda não temos uma soma de quadrados.

$$q(v) = (\sqrt{6}\, a)^2 - (\sqrt{4}\, b)^2$$

Base B' tal que se $[v]_B = \begin{bmatrix} a \\ b \end{bmatrix}$ então $[v]_{B'} = \begin{bmatrix} \sqrt{6}\, a \\ \sqrt{4}\, b \end{bmatrix}$

$$B' = \left\{ \frac{1}{\sqrt{6}} v_1 , \frac{1}{\sqrt{4}} v_2 \right\} \Rightarrow \boxed{q(v) = r^2 - s^2}$$

onde $[v]_{B'} = \begin{bmatrix} r \\ s \end{bmatrix}$

$$\begin{cases} r = \sqrt{6}\, a \\ s = \sqrt{4}\, b \end{cases} \Rightarrow \begin{cases} r = \sqrt{6}\, \frac{1}{\sqrt{2}} (x - y) \\ s = \sqrt{4}\, \frac{1}{\sqrt{2}} (x + y) \end{cases}$$

Matriz mudança de base:

$$B = \left\{ \frac{1}{\sqrt{2}} (1, -1) , \frac{1}{\sqrt{2}} (1, 1) \right\}.$$

$$P = [I]_C^B = \begin{bmatrix} [v_1]_C & [v_2]_C \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}_{} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

matriz ORTOGONAL

$$[I]_B^C = P^{-1} = P^T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

matriz de rotação

$$\begin{bmatrix} \cos \frac{-\pi}{4} & -\operatorname{sen}\left(-\frac{\pi}{4}\right) \\ \operatorname{sen} \frac{-\pi}{4} & \cos \frac{-\pi}{4} \end{bmatrix}$$

$$[v]_C = [I]_C^B [v]_B$$

$$[v]_B = [I]_B^C [v]_C \longrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\begin{cases} a = \frac{1}{\sqrt{2}} (x - y) \\ b = \frac{1}{\sqrt{2}} (x + y) \end{cases} \leftarrow$$

Verificando:

$$q(v) = 6a^2 - 4b^2$$

$$= 6\left(\frac{1}{\sqrt{2}}(x-y)\right)^2 - 4\left(\frac{1}{\sqrt{2}}(x+y)\right)^2$$

$$= 6\,\frac{1}{2}(x-y)^2 - 4\,\frac{1}{2}(x+y)^2$$

$$= 3(x^2 - 2xy + y^2) - 2(x^2 + 2xy + y^2)$$

$$= 3x^2 - 6xy + 3y^2 - 2x^2 - 4xy - 2y^2$$

$$q(v) = x^2 - 10xy + y^2$$

$$q(v) = r^2 - s^2 \leftarrow$$

$$\begin{cases} r = \sqrt{6}\,\frac{1}{\sqrt{2}}(x-y) \\ s = \sqrt{4}\,\frac{1}{\sqrt{2}}(x+y) \end{cases}$$

$$= \left(\sqrt{\tfrac{6}{2}}(x-y)\right)^2 - \left(\sqrt{\tfrac{4}{2}}(x+y)\right)^2$$

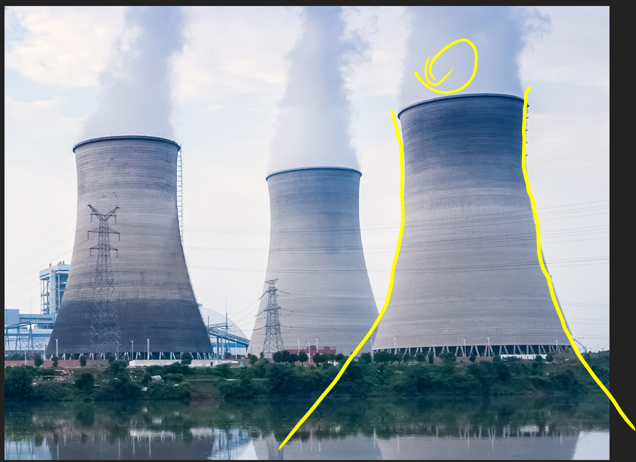
$$= 3(x-y)^2 - 2(x+y)^2$$

$$= x^2 - 10xy + y^2$$

$$[q]_B = [I]_B^C\,[q_C]\,[I]_C^B$$

$$\boxed{D = P^{-1} A P}$$

$$\begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$



hiperboloide de uma folha

$$Ax^2+By^2+Cz^2+Dxy+Exz+Fyz+Hx+Jy+Kz=L.$$

$$\lambda_1 a^2+\lambda_2 b^2+\lambda_3 c^2=f$$

$$\lambda_1,\lambda_2>0$$
$$\lambda_3<0$$