Notas de aula ECT2202 T03 2022-01-27 Aula 20 — Diagonalização

Lista da aula 20

3. Sejam a base canônica de $\mathbb{R}^3$: $B = \{(1,0,0), (0,1,0), (0,0,1)\}$, a base $C = \{(0,1,1), (0,-1,1), (1,0,1)\}$ de $\mathbb{R}^3$ e o operador linear $T \in L(\mathbb{R}^3)$ dado por

$$[T]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}.$$

(a) Encontre o polinômio característico de T, os autovalores de T e os autovetores correspondentes.

(b) Encontre $[T]_C$ e o polinômio característico. O que você pode dizer a respeito dele?

(c) Encontre uma base D de $\mathbb{R}^3$, se possível, tal que $[T]_D$ seja diagonal. Escreva a matriz M tal que $[T]_D = M^{-1}\,[T]_B\,M$.

$$[T]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \qquad B = \text{canônica}$$

$$[T(x,y,z)]_B = \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}}_{[T_B]} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_B$$

$$[T(x,y,z)]_C = [T]_C\,[(x,y,z)]_C$$

Matriz mudança de base:

$$[v]_C = [I]_C^B\,[v]_B \quad > \quad [I]_C^B = \left([I]_B^C\right)^{-1}$$

$$[v]_B = [I]_B^C\,[v]_C$$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$, a base $C = \{(0,1,1), (0,-1,1), (1,0,1)\}$

$$[I]_B^C = \left[\, [(0,1,1)]_B,\ [(0,-1,1)]_B,\ [(1,0,1)]_B \,\right]$$

$$[I]_B^C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = M$$

$$\to \left[T(x,y,z)\right]_B = [T]_B \left[(x,y,z)\right]_B \qquad \text{Tenho}$$

$$\left[T(x,y,z)\right]_C = [T]_C \left[(x,y,z)\right]_C \qquad \text{Quero}$$

Matriz mudança de base:

$$\begin{cases} [v]_C = [I]^B_C \, [v]_B \\ [v]_B = [I]^C_B \, [v]_C \end{cases} \Rightarrow [I]^B_C = \left([I]^C_B\right)^{-1}$$

$$[T(v)]_B = [T]_B \, [v]_B$$

$$[T(v)]_B = [T]_B \left([I]^C_B \, [v]_C\right)$$

$$\left([I]^C_B \, [T(v)]_C\right) = [T]_B \, [I]^C_B \, [v]_C$$

$$\left(\text{multiplicar } \left([I]^C_B\right)^{-1}\right)$$

$$\underbrace{\left([I]^C_B\right)^{-1} [I]^C_B}_{1} \, [T(v)]_C = \left([I]^C_B\right)^{-1} [T]_B \, \underbrace{[I]^C_B}_{M} \, [v]_C \qquad \to M = [I]^C_B$$

$$[T(v)]_C = \underbrace{M^{-1} [T]_B M} \, [v]_C$$

$$[T(v)]_C = [T]_C \, [v]_C$$

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$, a base $C = \{(0,1,1), (0,-1,1), (1,0,1)\}$

$$[I]_B^C = \left[[(0,1,1)]_B , [(0,-1,1)]_B , [(1,0,1)]_B \right]$$

$$[I]_B^C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = M$$

$$[I]_C^B = \left([I]_B^C \right)^{-1} \leftarrow$$

$$B = \{ \underset{v_1}{(1,0,0)} , \underset{v_2}{(0,1,0)} , \underset{v_3}{(0,0,1)} \}$$

$$C = \{ (0,1,1) , (0,-1,1) , (1,0,1) \}$$

$$[I]_C^B = \left[[(1,0,0)]_C \quad [(0,1,0)]_C \quad [(0,0,1)]_C \right]$$

$$(1,0,0) = \alpha(0,1,1) + \beta(0,-1,1) + \gamma(1,0,1)$$

$$\begin{cases} \gamma = 1 \\ \alpha - \beta = 0 \Rightarrow \alpha = \beta \\ \alpha + \beta + \gamma = 0 \Rightarrow 2\alpha + 1 = 0 \Rightarrow \alpha = \beta = -1/2 \end{cases}$$

$$[(1,0,0)]_C = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$(0,1,0) = \alpha(0,1,1) + \beta(0,-1,1) + \gamma(1,0,1)$$

$$\begin{cases} \gamma = 1 \\ \alpha - \beta = 1 \quad \Rightarrow \quad \alpha = 1+\beta \\ \alpha + \beta + \gamma = 0 \quad 1 + 2\beta + 1 = 0 \Rightarrow 2\beta = -2, \ \beta = -1 \end{cases}$$

$$[(0,1,0)]_C = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$(0,0,1) = \alpha(0,1,1) + \beta(0,-1,1) + \gamma(1,0,1)$$

$$\begin{cases} \gamma = 0 \\ \alpha - \beta = 0 \quad \Rightarrow \quad \alpha = \beta \\ \alpha + \beta + \gamma = 1 \quad \Rightarrow \quad 2\beta = 1 \Rightarrow \alpha = \beta = 1/2 \end{cases}$$

$$[(0,0,1)]_C = \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

$$[I]_C^B = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & -1 & 1/2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\det(M^{-1} T M) = \det(T)$$

$$\det M^{-1} \det T \det M$$

$$\frac{1}{\cancel{\det M}} \det T \ \cancel{\det M}$$

$$[T]_C = [I]_C^B \, [T]_B \, [I]_B^C$$

$$[T(v)]_C = [I]_C^B \, [T]_B \, [I]_B^C \, [v]_C$$

$$[T]_C = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & -1 & 1/2 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ -1 & -2 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ -2 & 4 & 1 \\ -3 & -3 & -3 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} -4 & -4 & -6 \\ 0 & -12 & -8 \\ -2 & 10 & 8 \end{bmatrix}$$

$$[T]_C = \begin{bmatrix} -2 & -2 & -3 \\ 0 & -6 & -4 \\ -1 & 5 & 4 \end{bmatrix}$$

Pol. carat.

$$\det([T]_C - \lambda I) \qquad \begin{bmatrix} -2-\lambda & -2 & -3 \\ 0 & -6-\lambda & -4 \\ -1 & 5 & 4-\lambda \end{bmatrix}$$

$$p(\lambda) = (-2-\lambda)\left[(-6-\lambda)(4-\lambda) + 20\right] - 1\left[8 - 3(6+\lambda)\right]$$

$$= (-2-\lambda)\left[-24 \underbrace{-4\lambda + 6\lambda}_{+2\lambda} + \lambda^2 + 20\right] - \left[8 - 18 - 3\lambda\right]$$

$$= -\lambda^3 - 2\lambda^2 + 24\lambda + 48 - 4\lambda - 2\lambda^2 - 40 - 20\lambda$$
$$- 8 + 18 + 3\lambda$$

$$p(\lambda) = -\lambda^3 - 4\lambda^2 + 3\lambda + 18 \quad //$$

$$p(2) = -8 - 4(4) + 3(2) + 18 \qquad \qquad p(-3) = -(-27) - 4(9) + 3(-3) + 18$$
$$= -24 + 18 + 6 = 0 \quad // \qquad \qquad = 27 - 36 - 9 + 18$$
$$= -9 - 9 + 18 = 0.$$

Note: $[T]_C = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & 0 & 1 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix}$$

$$= (2-\lambda)\left((-3-\lambda)^2\right)$$

raízes: 2 e -3.

$$= (2-\lambda)\left(9+6\lambda+\lambda^2\right)$$

$$= 18 + 12\lambda - \lambda^2 - 9\lambda - 3\lambda^2 - \lambda^3$$

$$p(\lambda) = -\lambda^3 - 4\lambda^2 + 3\lambda + 18 \quad //$$

λ = letra grega LAMBDA

por

$$[T]_B = \begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

(c) Encontre uma base D de $\mathbb{R}^3$, se possível, tal que $[T]_D$ seja diagonal. Escreva a matriz M tal que $[T]_D = M^{-1}[T]_B M$.

Autovalores 2 e -3 (duplo)

Autovetores

$\lambda = 2$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ tal que $[T]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

EQUAÇÃO DE AUTOVALORES

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2-2 & 0 & 1 \\ 0 & -3-2 & 1 \\ 0 & 0 & -3-2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} 2x + z = 2x \\ -3y + z = 2y \\ -3z = z \end{cases} \Rightarrow \begin{cases} z = 0 \\ -5y + z = 0 \\ -4z = 0 \end{cases} \Rightarrow \begin{matrix} z = 0 \\ y = 0 \\ x = \text{livre} \end{matrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B$$

B = canônica.

$(1,0,0)$ é autovetor de T com autovalor 2.

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$D = \{(1,0,0), \ldots, \ldots\}$.

$\lambda = -3$

$$\begin{bmatrix}2&0&1\\0&-3&1\\0&0&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = -3\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\begin{cases}2x + z = -3x\\ -3y + z = -3y\\ \cancel{-3z = -3z}\end{cases} \Rightarrow \begin{cases}5x + z = 0\\ z = 0\end{cases}$$

y : variável livre

$z = 0$, $5x = 0 \Rightarrow x = 0$

$$\begin{bmatrix}0\\y\\0\end{bmatrix} = y\begin{bmatrix}0\\1\\0\end{bmatrix}$$ é autovetor de T.

$$\dim(V(\lambda=-3)) = 1$$

A MULTIPLICIDADE GEOMÉTRICA
DO AUTOVALOR (λ = -3) É 1

$$p(\lambda) = (2-\lambda)(-3-\lambda)^2$$

A MULTIPLICIDADE ALGÉBRICA
DA RAIZ (λ=-3) NO POL. CARACRÍSTICO É 2

$$\begin{bmatrix}2&0&1\\0&-3&1\\0&0&-3\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\-3\\0\end{bmatrix} = -3\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$D = \{\underbrace{(1,0,0)}_{v_{(\lambda=2)}}, \underbrace{(0,1,0)}_{v_{(\lambda=-3)}}, \ldots\}$$

nÃO TEMOS AUTOVETORES L.I. SUFICIENTES PARA CONSTRUIR UMA BASE
LOGO T NÃO É DIAGONALIZÁVEL.