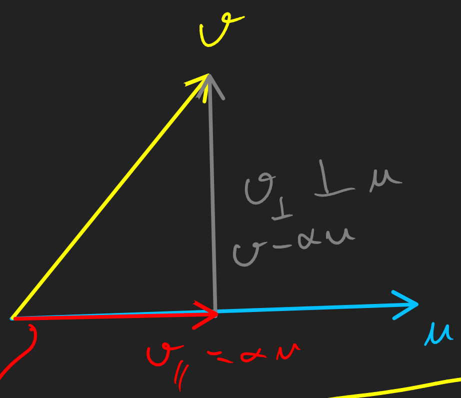
Nota de aulas ECT2202 — 2021-11-23 — T3 — BASES ORTONORMAIS

Projeção



$$\boxed{\mathrm{proj}_u v = \frac{\langle v, u\rangle}{\langle u, u\rangle} u}$$

$$v = v_\parallel + v_\perp \Rightarrow v_\perp = v - v_\parallel$$
$$= v - \alpha u$$

$$v_\parallel = \alpha u = \mathrm{proj}_u v$$

$$\langle v_\perp, u\rangle = 0$$

$$\langle v - \alpha u, u\rangle = 0$$

$$\langle v, u\rangle - \alpha\langle u, u\rangle = 0$$

$$\langle v, u\rangle = \alpha\langle u, u\rangle$$

COEFICIENTE DE FOURIER $\rightarrow$ $\dfrac{\langle v, u\rangle}{\langle u, u\rangle} = \alpha$

Notação

$\mathrm{proj}_u v$ $\qquad$ $\log_{10} x$

$\mathrm{proj}_u v = \dfrac{\langle v, u\rangle}{\langle u,u\rangle} u$

$$v_\perp = v - v_\parallel = v - \mathrm{proj}_u v$$

$$\boxed{v_\perp = v - \frac{\langle v, u\rangle}{\langle u, u\rangle} u}$$

Bases ortonormais

$$B = \{ v_1, v_2, \dots, v_n \}$$ ortonormal

Ortogonal: $\langle v_i, v_j \rangle = 0 \quad , i \neq j$

Normal (normalizada) $\| v_i \| = 1 \Rightarrow \| v_i \|^2 = 1$

$$\langle v_i, v_i \rangle = 1$$

Ortonormal

$$\boxed{\langle v_i, v_j \rangle = \delta_{ij} = \begin{cases} 0 & \text{se } i \neq j \\ 1 & \text{se } i = j \end{cases}}$$

Seja $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$.

$$[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$\langle v, v_1 \rangle = \langle \alpha_1 v_1 + \dots + \alpha_n v_n , v_1 \rangle$$

$$= \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_2, v_1 \rangle + \dots + \alpha_n \langle v_n, v_1 \rangle$$

$$= \alpha_1 + 0 + \dots + 0$$

$$\alpha_1 = \langle v, v_1 \rangle$$

$$\alpha_2 = \langle v, v_2 \rangle$$

$$\vdots$$

Ex.: $\mathbb{R}^3$ 3 elem.

$$B = \{ \underset{v_1}{(1,2,1)}, \underset{v_2}{(2,1,-4)}, \underset{v_3}{(3,-2,1)} \}$$

a) Mostre que B é uma base ortogonal de $\mathbb{R}^3$ mas não ortonormal

b) Escreva as coordenadas de (5, 3, 2) na base B

(produto interno ususal)

a) (Note: todo conjunto ortogonal é L.I.)

Mostrar ortogonalidade

$$\langle v_1, v_2 \rangle = \langle (1,2,1), (2,1,-4) \rangle = 2+2-4=0$$

$$\langle v_1, v_3 \rangle = \langle (1,2,1), (3,-2,1) \rangle = 3-4+1=0$$

$$\langle v_2, v_3 \rangle = \langle (2,1,-4), (3,-2,1) \rangle = 6-2-4=0 \checkmark$$

B é ortogonal, logo L.I. Como é L.I. com 3 elementos de $\mathbb{R}^3$, é base de $\mathbb{R}^3$.

Testar normas:

$$\|v_1\|^2 = \langle v_1, v_1 \rangle = 1^2+2^2+1^2 = 6 \neq 1$$

$$\|v_2\|^2 = \langle v_2, v_2 \rangle = 2^2+1^2+(-4)^2 = 21 \neq 1$$

$$\|v_3\|^2 = \langle v_3, v_3 \rangle = 3^2+(-2)^2+1^2 = 14 \neq 1$$

Portanto a base ortogonal B não é normalizada.

A base $B' = \left\{ \frac{1}{\sqrt{6}}(1,2,1), \frac{1}{\sqrt{21}}(2,1,-4), \frac{1}{\sqrt{14}}(3,-2,1) \right\}$

é orotonormal.

b) Coordenadas de v = (5, 3, -2) na base B

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

Como B é ortogonal

$$\alpha_1 = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle} ;$$

$$[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}_B$$

$$B = \{ \underset{v_1}{(1,2,1)}, \underset{v_2}{(2,1,-4)}, \underset{v_3}{(3,-2,1)} \}$$

$$\alpha_1 = \frac{\langle (5,3,-2), (1,2,1) \rangle}{\langle (1,2,1), (1,2,1) \rangle} = \frac{5+6-2}{6} = \frac{9}{6} = \frac{3}{2}$$

$$\alpha_2 = \frac{\langle (5,3,-2), (2,1,-4) \rangle}{\|(2,1,-4)\|^2} = \frac{10+3+8}{21} = \frac{21}{21} = 1$$

$$\alpha_3 = \frac{\langle (5,3,-2), (3,-2,1) \rangle}{\|(3,-2,1)\|^2} = \frac{15-6-2}{14} = \frac{7}{14} = \frac{1}{2}$$

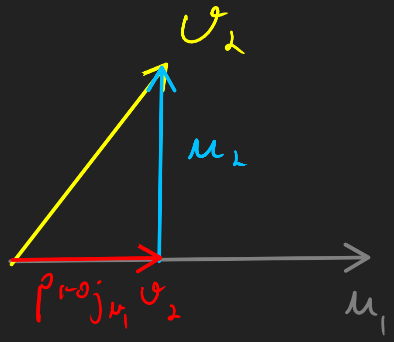
$$[(5,3,-2)]_B = \begin{bmatrix} 3/2 \\ 1 \\ 1/2 \end{bmatrix}_B$$

Processo de ortogonalização de Gram-Schmidt

Teorema: Todo espaço vetorial de dimensão finita possui base ortonormal.

Exemplo: $\mathbb{R}^3$:

$$B = \{\underset{v_1}{(1,0,1)}, \underset{v_2}{(0,1,1)}, \underset{v_3}{(0,0,1)}\}$$



Obter uma base ortonormal de $\mathbb{R}^3$ a partir de B.

$C = \{u_1, u_2, u_3\}$ ortogonal

$G = \left\{\frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|}\right\}$ ortonormal

$$u_1 = v_1 = (1,0,1)$$

$$u_2 = v_2 - proj_{u_1} v_2 = v_2 - \frac{\langle v_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$= (0,1,1) - \frac{\langle (0,1,1), (1,0,1)\rangle}{\langle (1,0,1), (1,0,1)\rangle}(1,0,1)$$

$$= (0,1,1) - \frac{1}{2}(1,0,1) = (-1/2, 1, 1/2)$$

$$u_3 = v_3 - proj_{u_2} v_3 - proj_{u_1} v_3$$

$$u_3 = v_3 - \frac{\langle v_3, u_2\rangle}{\langle u_2, u_2\rangle} u_2 - \frac{\langle v_3, u_1\rangle}{\langle u_1, u_1\rangle} u_1$$

$$= (0,0,1) - \frac{\langle (0,0,1), (-1/2, 1, 1/2)\rangle}{\|(-1/2, 1, 1/2)\|^2}(-1/2, 1, 1/2) - \frac{\langle (0,0,1), (1,0,1)\rangle}{\langle (1,0,1), (1,0,1)\rangle}(1,0,1)$$

$$= (0,0,1) - \frac{1/2}{1/4 + 1 + 1/4}(-1/2, 1, 1/2) - \frac{1}{2}(1,0,1)$$

$$= (0,0,1) - \frac{1}{2}\frac{2}{3}(-1/2, 1, 1/2) - (1/2, 0, 1/2)$$

$$= (0,0,1) - \frac{1}{6}(-1,2,1) - \frac{1}{2}(1,0,1)$$

$$= \left(0 + \frac{1}{6} - \frac{1}{2},\ 0 - \frac{2}{6} + 0,\ 1 - \frac{1}{6} - \frac{1}{2}\right)$$

$$= \left(-1/3,\ -\frac{1}{3},\ \frac{1}{3}\right)$$

$$C = \left\{ (1,0,1),\ (-1/2, 1, 1/2),\ (-1/3, -1/3, \frac{1}{3}) \right\} //$$

C é ortogonal.

Normalizar:

$$G = \left\{ \frac{(1,0,1)}{\sqrt{\langle (1,0,1), (1,0,1) \rangle}},\ \frac{(-1/2, 1, 1/2)}{\sqrt{3/2}},\ \frac{(-1/3, -1/3, \frac{1}{3})}{\sqrt{3/9}} \right\}$$

$$G = \left\{ \frac{1}{\sqrt{2}}(1,0,1),\ \frac{1}{\sqrt{6}}(-1,2,1),\ \frac{1}{\sqrt{3}}(-1,-1,1) \right\} //$$

é base ortonormal de $\mathbb{R}^3$.

$P_2(\mathbb{R})$ $\quad B = \{1, t, t^2\}$ $\quad C = (u_1, u_2, u_3)$

$$\langle f, g \rangle = \int_{-1}^{1} f(t) g(t)\, dt$$

$C = (1,$

$u_1 = 1$

$$u_2 = t - \frac{\langle t, 1 \rangle}{\langle 1, 1 \rangle} 1 = t - \frac{\int_{-1}^{1} t\, dt}{\int_{-1}^{1} 1\, dt} 1 = t //$$

$$u_2 = t^2 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle}\, t - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle}\, 1$$

$$= t^2 - \frac{\cancelto{0}{\int_{-1}^{1} t^3\, dt}}{\int_{-1}^{1} t^2\, dt}\, t - \frac{\int_{-1}^{1} t^2\, dt}{\int_{-1}^{1} 1\, dt}\, 1$$

$$= t^2 - \frac{2/3}{2}\, 1 = t^2 - \frac{1}{3} \,/\!/$$

$$C = \{1, t, t^2 - 1/3\}$$

POLINÔMIOS DE LEGENDRE