Notas de aula ECT2202 T02 2021-11-18 Aulas 07-08 — Produto interno, norma...

Definição	No caso complexo
Seja V um espaço vetorial sobre $\mathbb{R}$. Um *produto interno* transforma cada par ordenado de vetores $(\boldsymbol{u}, \boldsymbol{v})$ de V em um único número real denotado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ e obedece aos seguintes axiomas:	Seja V um espaço vetorial sobre $\mathbb{C}$.

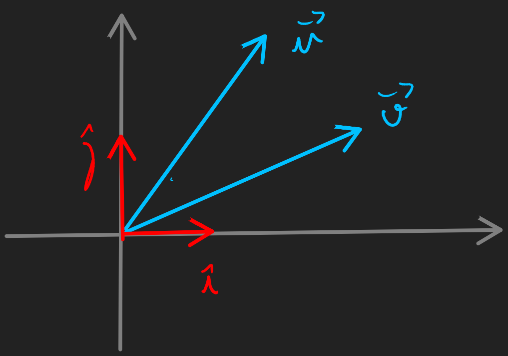
Definição

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
3. $\langle \boldsymbol{u}, \alpha \boldsymbol{v} \rangle = \alpha \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
4. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0$, sendo que $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$

No caso complexo

- O produto interno é um número complexo.
- O Axioma 1 passa a ser $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$

Produto escalar



Definição (no espaço da geometria)

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(\theta)$$

$$\vec{u} = x\,\hat{\imath} + y\,\hat{\jmath} \Rightarrow [\vec{u}] = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\vec{v} = a\,\hat{\imath} + b\,\hat{\jmath} \Rightarrow [\vec{v}] = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = (x\hat{\imath} + y\hat{\jmath}) \cdot (a\hat{\imath} + b\hat{\jmath})$$

$$= xa\,\hat{\imath}\cdot\hat{\imath} + xb\,\hat{\imath}\cdot\hat{\jmath} + ya\,\hat{\jmath}\cdot\hat{\imath} + yb\,\hat{\jmath}\cdot\hat{\jmath}$$

$$= xa + yb$$

$\hat{\imath}\cdot\hat{\imath} = \|\hat{\imath}\|\|\hat{\imath}\|\cos 0 = 1$

$\hat{\imath}\cdot\hat{\jmath} = \|\hat{\imath}\|\|\hat{\jmath}\|\cos \frac{\pi}{2} = 0$

Exemplos

$\mathbb{R}^3$: produto interno usual

$u = (x, y, z)$

$v = (a, b, c)$

$\langle u, v \rangle = xa + yb + zc$

$\langle v, u \rangle = ax + by + cz$

iguais (Axioma 1)

No $\mathbb{R}^2$: Produto interno (não usual)

$$u = (x, y) \qquad v = (a, b) \qquad \langle u, v \rangle = xa - xb - ya + 3yb$$

$$u = (1, 3) \qquad v = (-2, 1)$$

$$\langle u, v \rangle = 1(-2) - 1(1) - 3(-2) + 3(3)(1) = -2 - 1 + 6 + 9 = 12$$

Prod. interno usual

$$\langle u, v \rangle = 1(-2) + 3(1) = -2 + 3 = 1$$

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Espaço vetorial das matrizes reais

$M_{2\times 2}(\mathbb{R})$:

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow B^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Seja o produto interno $\langle A, B \rangle = tr(B^t A)$

$$\langle A, B \rangle = tr\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} \right) = tr \begin{pmatrix} ax + cz & \\ & by + dw \end{pmatrix}$$

$$= (ax + cz) + (by + dw)$$

$$= ax + by + cz + dw$$

$$\left\{ \underset{E_1}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}, \underset{E_2}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \underset{E_3}{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}, \underset{E_4}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} \right\}$$

$$A = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = xE_1 + yE_2 + zE_3 + wE_4$$

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = aE_1 + bE_2 + cE_3 + dE_4$$

$$\langle A, B\rangle = \langle xE_1 + yE_2 + zE_3 + wE_4,\ aE_1 + bE_2 + cE_3 + dE_4\rangle$$

$$= xa\underbrace{\langle E_1, E_1\rangle}_{1} + yb\underbrace{\langle E_2, E_2\rangle}_{1} + zc\underbrace{\langle E_3, E_3\rangle}_{1} + wd\underbrace{\langle E_4, E_4\rangle}_{1}$$

$$+ xb\underbrace{\langle E_1, E_2\rangle}_{0} + xc\underbrace{\langle E_1, E_3\rangle}_{0} + xd\underbrace{\langle E_1, E_4\rangle}_{0}$$

$$+ ya\underbrace{\langle E_2, E_1\rangle}_{0} + yc\underbrace{\langle E_2, E_3\rangle}_{0} + yd\underbrace{\langle E_2, E_4\rangle}_{0}$$

$$+ za\underbrace{\langle E_3, E_1\rangle}_{0} + zb\underbrace{\langle E_3, E_2\rangle}_{0} + zd\underbrace{\langle E_3, E_4\rangle}_{0}$$

$$+ wa\underbrace{\langle E_4, E_1\rangle}_{0} + wb\underbrace{\langle E_4, E_2\rangle}_{0} + wc\underbrace{\langle E_4, E_3\rangle}_{0}$$

$$\langle E_1, E_1\rangle = tr\left(E_1^t E_1\right) = tr\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = tr\begin{pmatrix}1&0\\0&0\end{pmatrix} = 1$$

$$= 1(1) + 0 + 0 + 0$$

$$\langle E_1, E_2\rangle = tr\left(E_2^t E_1\right) = tr\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = tr\begin{pmatrix}0&0\\1&0\end{pmatrix} = 0$$

$$\left\langle \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}\right\rangle = 0(1) + 1(0) + 0 + 0 = 0.$$

$$\left\{ \underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix}0&1\\0&0\end{bmatrix}}_{E_2}, \underbrace{\begin{bmatrix}0&0\\1&0\end{bmatrix}}_{E_3}, \underbrace{\begin{bmatrix}0&0\\0&1\end{bmatrix}}_{E_4} \right\}$$

$$\langle E_i, E_j\rangle = \underbrace{\delta_{ij}}_{\text{delta de Kronecker}} = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$$\langle E_1, E_1\rangle = 1 \quad ; \quad \langle E_1, E_2\rangle = 0$$

$M_{2\times3}$

$$A = \begin{pmatrix} \boxed{9 \ 8 \ 7} \\ 6 \ 5 \ 4 \end{pmatrix} \qquad B = \begin{pmatrix} \boxed{1 \ 2 \ 3} \\ 4 \ 5 \ 6 \end{pmatrix}$$

$$\langle A, B \rangle = tr(B^t A) = tr\left(\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \end{bmatrix} \right)$$

$$= 9(1) + 8(2) + 7(3) + 6(4) + 5(5) + 6(4) = 119$$

$$\langle A, B \rangle = [1 \ 2 \ 3] \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} + [4 \ 5 \ 6] \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$$

Base canônica $M_{2\times3}$:

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_1}, \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_2}, \ldots, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_6} \right\}.$$

$$\langle E_i, E_j \rangle = \delta_{ij}$$

BASE ORTONORMAL

· Espaço vetorial das funções contínuas em [-1, 1]

$$\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$$

$f(t) = t^2 - 1$

$g(t) = t$

$\langle f, g \rangle = 0 \ \therefore \ f \perp g$

ortogonal "perpendicular"

$$\langle f, g \rangle = \int_{-1}^{1} (t^2 - 1)\, t \, dt = \int_{-1}^{1} t^3 - t \, dt = 0$$

integral simétrica de função ímpar

$$\left[\frac{t^4}{4} - \frac{t^2}{2} \right]_{-1}^{1} = \left[\frac{1^4}{4} - \frac{1^2}{2} \right] - \left[\frac{(-1)^4}{4} - \frac{(-1)^2}{2} \right] = 0$$

Espaço vetorial das funções conínuas em [0,1]

$$\langle f,g\rangle = \int_0^1 f(t)\,g(t)\,dt$$

$f = t^2-1$ $\qquad \langle f,g\rangle = \int_0^1 (t^2-1)t\,dt = \int_0^1 t^3 - t\,dt$

$g = t$

$$\left[\frac{t^4}{4} - \frac{t^2}{2}\right]_0^1 = \frac{1}{4} - \frac{1}{2} - [0] = -\frac{1}{2} \neq 0$$

Norma

Definição

Seja V um espaço vetorial dotado de produto interno. A *norma* do vetor $\boldsymbol{v}$ de V é o número real positivo dado por

$$\boxed{\|\boldsymbol{v}\| = \sqrt{\boxed{\langle \boldsymbol{v}, \boldsymbol{v}\rangle}}}$$

$\langle f,g\rangle = \int_{-1}^1 f(t)\,g(t)\,dt \qquad f = t^2-$

$$\|f\| = \sqrt{\langle f, f\rangle}$$

$$\langle f,f\rangle = \int_{-1}^1 (t^2-1)^2\,dt = \int_{-1}^1 t^4 - 2t^2 + 1\,dt$$

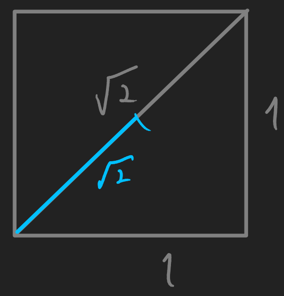
$$\left[\frac{t^5}{5} - 2\frac{t^3}{3} + t\right]_{-1}^1 = \left[\frac{1}{5} - \frac{2}{3} + 1\right] - \left[-\frac{1}{5} + \frac{2}{3} - 1\right]$$

$$= \frac{2}{5} - \frac{4}{3} + 2 = \frac{6-20}{15} + 2$$

$$\langle f,f\rangle = -\frac{14}{15} + 2 = \frac{16}{15}$$

$$\|f\| = \sqrt{\frac{16}{15}} = \frac{4}{\sqrt{15}}$$

$$\frac{1}{\sqrt{2}} \qquad \frac{\sqrt{2}}{2}$$



$$\|f\|^2 = \langle f, f\rangle$$

Espaço vetorial $\mathbb{R}^2$ prod. interno usual

$\|2u+2v\|^2 \qquad u=(1,-1) \qquad v=(1,0)$

$$\begin{aligned}
\|2u+2v\|^2 &= \langle 2u+2v, 2u+2v\rangle \\
&= \langle 2u, 2u+2v\rangle + \langle 2v, 2u+2v\rangle \\
&= \langle 2u, 2u\rangle + \langle 2u, 2v\rangle + \langle 2v, 2u\rangle + \langle 2v, 2v\rangle \\
&= 4\langle u,u\rangle + 4\langle u,v\rangle + 4\langle v,u\rangle + 4\langle v,v\rangle \\
&= 4\langle u,u\rangle + 8\langle u,v\rangle + 4\langle v,v\rangle \\
&= 4(1^2+1^2) + 8(1(1)+(-1)0) + 4(1(1)+0(0)) \\
&= 4(2) + 8 + 4 = 20.
\end{aligned}$$

$$\begin{aligned}
\|2u+2v\|^2 &= \|2(1,-1)+2(1,0)\|^2 \\
&= \|(4,-2)\|^2 = \langle (4,-2), (4,-2)\rangle \\
&= 16+4 = 20.
\end{aligned}$$