Notas de aula ECT2202 T02 2022-01-27 Aula 20 — Diagonalização

Seja A ∈ L(ℝ³) operador linear.

Na base canônica

Simétrica

$$[A]_C = \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$C = \{(1,0,0), (0,1,0), (0,0,1)\}$

$A(x,y,z) = (2x+2y, 2x-y, 2z)$

$$[A(x,y,z)]_C = [A]_C \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Obtenha os autovalores e autovetores de A

λ tal que $\underbrace{\det(A - \lambda I)}_{p_A(\lambda)} = 0$

$$\det\begin{pmatrix} 2-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$= (2-\lambda)\left[(2-\lambda)(-1-\lambda) - 4\right]$$

$$= (2-\lambda)\left[\lambda^2 - 2\lambda + \lambda - 2 - 4\right]$$

$$p_A(\lambda) = (2-\lambda)\left[\lambda^2 - \lambda - 6\right]$$

Raízes. $(2-\lambda) = 0 \Rightarrow \lambda = 2$

$\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = \begin{cases} 3 \\ -2 \end{cases}$

Autovalores: $\lambda_1 = 2 \Rightarrow V(\lambda_1)$ subespaço de autovetores

$\lambda_2 = 3 \Rightarrow V(\lambda_2)$ subespaço de autovetores

$\lambda_3 = -2 \Rightarrow V(\lambda_3)$ subespaço de autovetores

Para cada λ: $\underline{A(v) = \lambda v}$ (eq. de autovalores)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ tal que $\longrightarrow$ $\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\lambda_1 = 2$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \overbrace{\begin{bmatrix} 2-2 & 2 & 0 \\ 0 & -1-2 & 0 \\ 0 & 0 & 2-2 \end{bmatrix}}^{A - \lambda I} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} 2x + 2y = 2x \\ 2x - y = 2y \\ 2z = 2z \end{cases} \Rightarrow \begin{cases} 2x - 2x + 2y = 0 \\ 2x - y - 2y = 0 \\ 2z - 2z = 0 \end{cases}$$

$$\begin{cases} 2y = 0 \\ 2x - 3y = 0 \\ 0 = 0 \end{cases} \quad \begin{cases} y = 0 \\ 2x - 3y = 0 \Rightarrow x = 0 \\ z \text{ var. } \underline{\text{livre}} \end{cases}$$

$\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ é solução.

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2z \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} \checkmark$$

$(0,0,1)$ é autovetor de A com autovalor 2.

$\lambda = 3$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{cases} 2x + 2y = 3x \\ 2x - y = 3y \\ 2z = 3z \end{cases}$$

$$\begin{cases} -x + 2y = 0 \\ 2x - 4y = 0 \\ -z = 0 \end{cases} \Rightarrow \begin{cases} 2y = x \\ 2(2y) - 4y = 0 \\ z = 0 \end{cases}$$

$$\begin{cases} x = 2y \\ z = 0 \\ y \end{cases}$$ variável livre.

$$\begin{bmatrix} 2y \\ y \\ 0 \end{bmatrix} = y\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$: $(2,1,0)$ é autovetor de A com autovalor 3.

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4+2 \\ 4-1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \checkmark$$

$\lambda = -2$

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{cases} 2x + 2y = -2x \\ 2x - y = -2y \\ 2z = -2z \end{cases}$$

$$\begin{cases} 4x + 2y = 0 \\ 2x + y = 0 \\ 4z = 0 \end{cases} \Rightarrow \begin{cases} 4x + 2(-2x) = 0 \\ y = -2x \\ z = 0 \end{cases}$$

$$\begin{cases} 4x - 4x = 0 \\ y = -2x \\ z = 0 \end{cases} \Rightarrow \begin{cases} 0 = 0 \\ y = -2x \\ z = 0 \end{cases}$$ x var. livre

vetores do tipo $\begin{bmatrix} x \\ -2x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ são autovetores de A com autovalor -2

$$\begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2-4 \\ 2+2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Seja D o conjunto

$$D = \{ \underset{\lambda=2}{(0,0,1)}, \underset{\lambda=3}{(2,1,0)}, \underset{\lambda=-2}{(1,-2,0)} \}$$ (ortogonal)

é L.I. com 3 elem. de $\mathbb{R}^3$, logo é <u>BASE</u> de $\mathbb{R}^3$.

$$D' = \left\{ (0,0,1),\ \frac{1}{\sqrt{5}}(2,1,0),\ \frac{1}{\sqrt{5}}(1,-2,0) \right\}$$ orto NORMAL.

Obter a matriz $[A]_D$:

$$[A(v)]_C = [A]_C [v]_C \qquad [v]_D = [I]_D^C [v]_C$$

$$[A(v)]_D = [A]_D [v]_D \qquad [v]_C = [I]_C^D [v]_D$$

$$[I]_D^C = \left([I]_C^D\right)^{-1}$$

$$[A(v)]_C = [A]_C [v]_C$$

$$[A(v)]_C = [A]_C \left([I]_C^D [v]_D\right)$$

$$[I]_C^D [A(v)]_D = [A]_C [I]_C^D [v]_D$$

multiplicar $\left([I]_C^D\right)^{-1}$

$$\cancel{\left([I]_C^D\right)^{-1} [I]_C^D} [A(v)]_D = \left([I]_C^D\right)^{-1} [A]_C [I]_C^D [v]_D$$

$$[A(v)]_D = \underbrace{\left([I]_C^D\right)^{-1} [A]_C [I]_C^D}_{[A]_D} [v]_D$$

$$[A(v)]_D = [A]_D \quad [v]_D$$

$$M = [I]_C^D$$

$$\boxed{[A]_D = M^{-1} [A]_C M}$$

mudança de base
de operador linear

$$[A(v)]_D = [A]_D [v]_D = [I]_D^C [A]_C [I]_C^D [v]_D$$

$$D = P^{-1} A P \to A = P D P^{-1}$$

$$[I]^{D}_{C} = \Big[\, [v(\lambda_1)]_C \quad [v(\lambda_2)]_C \quad [v(\lambda_3)]_C \,\Big]$$

$$[I]^{D}_{C} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}$$

Na base ortonormal:

$$M = [I]^{D'}_{C} = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ \sqrt{5} & 0 & 0 \end{bmatrix}$$

é ortogonal

$$[I]^{C}_{D'} = \left([I]^{D'}_{C}\right)^{-1} = \left([I]^{D'}_{C}\right)^{T}$$

$$M^{-1} = [I]^{C}_{D'} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ \sqrt{5} & 0 & 0 \end{pmatrix}\right) = \frac{1}{5} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[A]_{D'} = M^{-1} [A]_C M$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ \sqrt{5} & 0 & 0 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 2 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 6 & -2 \\ 0 & 3 & 4 \\ 2\sqrt{5} & 0 & 0 \end{bmatrix}$$

$$[A]_D = \frac{1}{5}\begin{bmatrix} 2\cdot 5 & 0 & 0 \\ 0 & 12+3 & 0 \\ 0 & 6-6 & -2-8 \\ & & -10 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$D = \left\{ \overset{v(\lambda_1)}{(0,0,1)}, \overset{v(\lambda_2)}{\frac{1}{\sqrt{5}}(2,1,0)}, \overset{v(\lambda_3)}{\frac{1}{\sqrt{5}}(2,-1,0)} \right\} = B$$

$$[A]_D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$M_C^D = \begin{bmatrix} [v_1]_C & [v_2]_C & [v_3]_C \end{bmatrix}$$

4. Seja $A \in L(\mathbb{R}^3)$ o operador linear cuja matriz relativa à base canônica é

$$[A] = \begin{bmatrix} 2 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(a) Obtenha os autovalores e autovetores de A.
(b) Mostre que existe uma base B de $\mathbb{R}^3$, ortonormal, formada por autovetores de A.
(c) Escreva a matriz mudança de base da base de autovetores de A para a base canônica. Verifique que é uma matriz ortogonal $\boxed{M^{-1} = M^t}$.
(d) Mostre que $[A]_B = M^t[A]M$.