Definição

Seja V um espaço vetorial sobre $\mathbb{R}$. Um *produto interno* transforma cada par ordenado de vetores $(\boldsymbol{u}, \boldsymbol{v})$ de V em um único número real denotado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ e obedece aos seguintes axiomas:

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
3. $\langle \boldsymbol{u}, \alpha \boldsymbol{v} \rangle = \alpha \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
4. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0$, sendo que $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$

No caso complexo

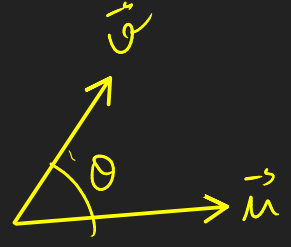
Seja V um espaço vetorial sobre $\mathbb{C}$.

- O produto interno é um número complexo.
- O Axioma 1 passa a ser $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$

$z = a + bi \Rightarrow \bar{z} = a - bi$

Em VGA: produto escalar:

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$



Norma
direção
sentido

No base $\{\hat{i}, \hat{j}, \hat{k}\}$ canônica:

$$[\vec{u}] = \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \quad [\vec{v}] = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$\vec{u} \cdot \vec{v} = [u]^T [v]$

$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$

$= \begin{bmatrix} u_x & u_y & u_z \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

No $\mathbb{R}^3$, $M_{3\times 1}(\mathbb{R})$

7.2.3. Exercício

1. Verifique que o produto interno usual sobre o $\mathbb{C}^n$ dado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} \overline{u_i} v_i$ satisfaz os axiomas de produto interno.

$$\langle u, v \rangle = \sum_{i=1}^{n} \bar{u}_i v_i = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n$$

Def. prod. interno.

Axiomas

1) $\langle v, u \rangle = \overline{\langle u, v \rangle}$ $\qquad \overline{\langle u, v \rangle} = \overline{\sum_i \bar{u}_i v_i}$

$$\langle v, u \rangle = \sum_{i=1}^{n} \bar{v}_i u_i \qquad u_i, \bar{v}_i \in \mathbb{C}$$

$$= \sum_i u_i \bar{v}_i \qquad \overline{(\bar{z})} = z$$

$$= \sum_i (\bar{u}_i) \bar{v}_i \qquad \bar{z}_1 \bar{z}_2 = \overline{(z_1 z_2)}$$

$$= \sum_i \overline{(\bar{u}_i v_i)} \qquad \bar{z}_1 + \bar{z}_2 = \overline{(z_1 + z_2)}$$

$$= \overline{\left(\sum_i \bar{u}_i v_i \right)} = \overline{\langle u, v \rangle}$$ ∎

2) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$$\langle u, v+w \rangle = \sum_i \bar{u}_i (v_i + w_i) \qquad : u_1(v_1+w_1) + u_2(v_2+w_2) \cdots$$

$$= \sum_i \left[\bar{u}_i v_i + \bar{u}_i w_i \right] \qquad (u_1 v_1 + u_1 w_1) + (u_2 v_2 + u_2 w_2)$$

$$u_1 v_1 + u_1 v_2 + \cdots + u_1 w_1 + u_1 w_2$$

$$= \sum_i \bar{u}_i v_i + \sum_i \bar{u}_i w_i$$

$$= \langle u, v \rangle + \langle u, w \rangle$$ ∎

3) $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$

$$\langle u, \alpha v \rangle = \sum_i \bar{u}_i (\alpha v_i) \qquad \bar{u}_1 \alpha v_1 + \bar{u}_2 \alpha v_2 \cdots$$

$$\alpha (\bar{u}_1 v_1 + \cdots)$$

$$= \alpha \sum_i \bar{u}_i v_i$$

$$= \alpha \langle u, v \rangle$$ ∎

4) $\langle u, u \rangle \geq 0$ e $\langle u, u \rangle = 0 \iff u = 0$

$\langle u, u \rangle \in \mathbb{R}$.

$$\langle u, u \rangle = \sum_i \bar{u}_i u_i$$

$$= \sum_i \underbrace{(\mathrm{Re}(u_i))^2}_{\geq 0} + \underbrace{(\mathrm{Im}(u_i))^2}_{\geq 0}$$

≥ 0. ∎

$z = a + bi$

$\bar{z} z = (a - bi)(a + bi)$

$a^2 - (bi)^2$

$a^2 - b^2 i^2$

$a^2 + b^2 \geq 0$ ($a^2 \geq 0$, $b^2 \geq 0$)

Produto interno não usual $\mathbb{R}^2$

$u = (u_1, u_2)$

$v = (v_1, v_2)$

$$\langle u, v \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + 5 u_2 v_2$$

Axiomas. Exercício.

1) $\langle v, u \rangle = \langle u, v \rangle$

$$\langle v, u \rangle = v_1 u_1 - v_1 u_2 - v_2 u_1 + 5 v_2 u_2$$

$$= u_1 v_1 - u_2 v_1 - u_1 v_2 + 5 u_2 v_2$$

$$= u_1 v_1 - u_1 v_2 - u_2 v_1 + 5 u_2 v_2$$

$$= \langle u, v \rangle$$

2) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ Exercício.

3) $\langle u, \alpha v \rangle = u_1 \alpha v_1 - u_1 \alpha v_2 - u_2 \alpha v_1 + 5 u_2 \alpha v_2$

$= \alpha \langle u, v \rangle.$

4) $\langle u, u \rangle \geq 0$

$\langle u, u \rangle = \underbrace{u_1 u_1}_{\geq 0} - u_1 u_2 - u_2 u_1 + 5 \underbrace{u_2 u_2}_{\geq 0}$

$= u_1^2 - 2 u_1 u_2 + 5 u_2^2$

$= (u_1^2 - 2 u_1 u_2 + u_2^2) + 4 u_2^2$

$= \underbrace{(u_1 - u_2)^2}_{\geq 0} + \underbrace{4 u_2^2}_{\geq 0} \geq 0.$

No caso $\mathbb{C}^n$:

Produto interno usual:

$$\langle u, v \rangle = \sum_i \bar{u}_i v_i$$

Na base canônica:

$$[u] = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad [v] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\langle u, v \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n$$

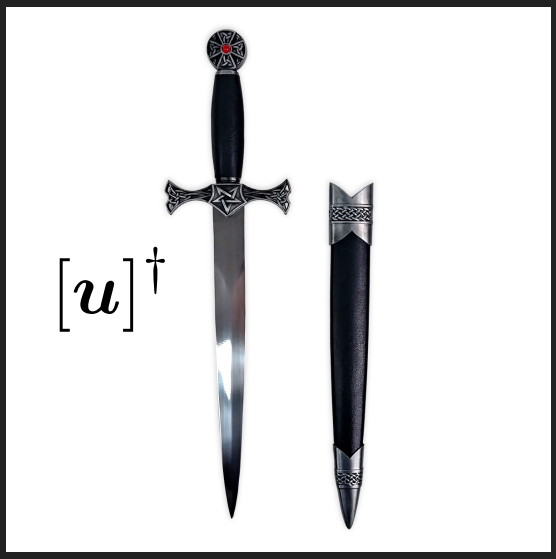
$$= \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \overline{\left([u]^T\right)}\,[v]$$

$$= [u]^\dagger [v]$$

$[u]^\dagger$ matriz HERMITEANA CONJUGADA de [u]: transposta complexa conjugada

$[u]^\dagger$ adoga dagger



Produto interno não usual $\mathbb{R}^2$

$u = (u_1, u_2)$
$v = (v_1, v_2)$

$$\langle u, v \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + 5 u_2 v_2$$

$$C = \{ e_1 = (1,0),\ e_2 = (0,1) \}$$

$$u = u_1 e_1 + u_2 e_2 \qquad v = v_1 e_1 + v_2 e_2$$

$$\langle u, v \rangle = \langle u_1 e_1 + u_2 e_2,\ v \rangle$$

$$= u_1 \langle e_1, v \rangle + u_2 \langle e_2, v \rangle$$

$$= u_1 \langle e_1, v_1 e_1 + v_2 e_2 \rangle + u_2 \langle e_2, v_1 e_1 + v_2 e_2 \rangle$$

$$= u_1 v_1 \langle e_1, e_1 \rangle + u_1 v_2 \langle e_1, e_2 \rangle +$$
$$+ u_2 v_1 \langle e_2, e_1 \rangle + u_2 v_2 \langle e_2, e_2 \rangle$$

$$= \begin{bmatrix} u_1 & v_1 \end{bmatrix} \begin{bmatrix} \langle e_1, e_1 \rangle & \langle e_1, e_2 \rangle \\ \langle e_2, e_1 \rangle & \langle e_2, e_2 \rangle \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\langle u, v \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + 5 u_2 v_2$$

Elementos de matriz

$$\langle e_1, e_1 \rangle = \langle (1,0), (1,0) \rangle =$$

$$= 1(1) - 1(0) - 0(1) + 5(0)(0) = 1$$

$$\langle e_1, e_2 \rangle = \langle e_2, e_1 \rangle$$

$$= \langle (1,0), (0,1) \rangle$$

$$= 1(0) - 1(1) - 0(0) + 5(0)(1)$$

$$= -1$$

$$\langle e_2, e_2 \rangle = \langle (0,1), (0,1) \rangle$$

$$= (0)0 - 0(1) - 1(0) + 5(1)(1) = 5.$$

$$\langle u, v \rangle = [u]_c^T \begin{bmatrix} 1 & -1 \\ -1 & 5 \end{bmatrix} [v]_c$$

Contra-exemplo: $\mathbb{R}^2$

$\langle u, v \rangle = u_1 v_1 u_2 v_2$ Não é prod. Int.

1) $\langle v, u \rangle = v_1 u_1 v_2 u_2 = u_1 v_1 u_2 v_2 = \langle u, v \rangle$ ✓

⋮

3) $\langle u, \alpha v \rangle = u_1 (\alpha v_1) u_2 (\alpha v_2)$

$= \alpha^2 u_1 v_1 u_2 v_2 \neq \alpha \langle u, v \rangle$ ✗

Espaço $P_2(\mathbb{R})$.

$$\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$$

é prod. interno.

$f = 2 + t \qquad g = -2 + 3t \qquad h = -3 - 2t + t^2$

$\langle f, g \rangle$, $\langle f, h \rangle$, $\langle f, 2g + 3h \rangle$.

$$\langle f, g \rangle = \int_0^1 (2+t)(-2+3t)\, dt$$

$$= \int_0^1 -4 + 6t - 2t + 3t^2\, dt$$

$$= \int_0^1 -4 + 4t + 3t^2\, dt$$

$$= \left[-4t + 2t^2 + t^3 \right]_0^1 = -4 + 2 + 1 = -1$$

$\langle f, g \rangle = -1$.

$$\langle f, h \rangle = \int_0^1 (2+t)(-3-2t+t^2)\,dt$$

$$\int_0^1 -6-4t+2t^2-3t-2t^2+t^3\,dt$$

$$\int_0^1 -6-7t+t^3\,dt$$

$$\left[-6t-\frac{7}{2}t^2+\frac{t^4}{4}\right]_0^1 = -6-\frac{7}{2}+\frac{1}{4}$$

$$= \frac{-24-14+1}{4} = \frac{-37}{4}.$$

$$\langle f, h \rangle = \frac{-37}{4}$$

$$\langle f, 2g+3h \rangle = \int_0^1 (2+t)\left[2(-2+3t)+3(-3-2t+t^2)\right]dt$$

$$\langle f, 2g+3h \rangle = 2\langle f, g \rangle + 3\langle f, h \rangle$$

$$= 2(-7) + 3\left(\frac{-37}{4}\right)$$

$$= -\frac{8}{4} - \frac{111}{4} = \frac{-119}{4} \;//$$

$\langle f, g \rangle$ si $f = f_1 v_1 + f_2 v_2 + \cdots$

$g = g_1 v_1 + \cdots$

$$\langle f, g \rangle = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \end{bmatrix} \begin{bmatrix} \langle e_1, e_1 \rangle & & & \\ & \ddots & & \\ & & & \end{bmatrix} \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}$$

Definimos o prod. interno:

$$\langle f, g \rangle = \int_a^b f(t)\, w(t)\, g(t)\, dt$$

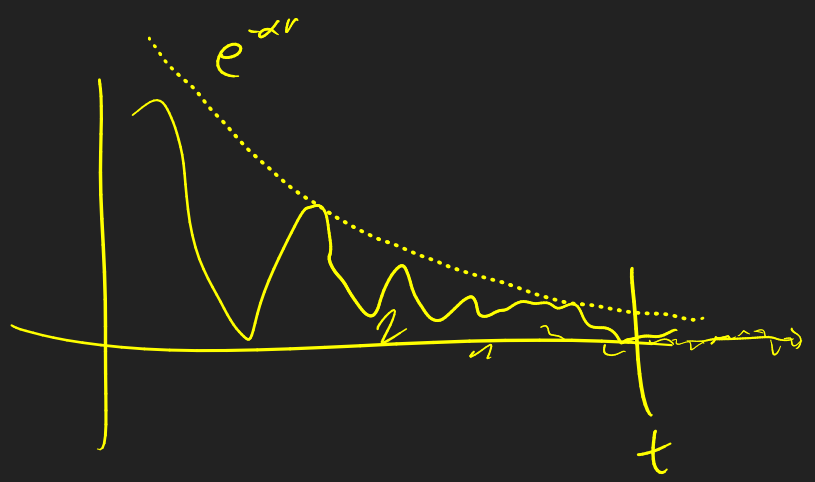
com $w(t)$ dado.

Ex.: Átomo de Hidrogenio:

orbitais ?, ?, ...

$$\langle f, g \rangle = \int_0^\infty f(r)\, \underbrace{e^{-ar}}_{\text{função peso.}}\, g(r)\, dr$$

$$\langle f, f \rangle = \int_0^\infty f(r)\, e^{-ar}\, f(r)\, dr = \text{existir}$$

(polinom.)

$$\lim_{t \to \infty} \int_0^t f^2(r)\, e^{-ar}\, dr$$