Orotogonalidade, norma

EXEMPLO $\mathbb{R}^2$ produto interno usual

$$u = (1, 5) \quad v = (3, 4) \quad w = (7, a)$$

a) Calcule $\langle u, v\rangle$, $\langle v, w\rangle$

b) Obtenha a t.q. $u \perp w$ $(\langle u, w\rangle = 0)$

c) Calcule $\|u + v - 2w\|$

a) $\langle u, v\rangle = \langle (1,5), (3,4)\rangle = 1 \cdot 3 + 5 \cdot 4 = 23$ ✓

$\langle v, w\rangle = \langle (3,4), (7,a)\rangle = 3 \cdot 7 + 4a = 21 + 4a$ ✓ //

b) $\boxed{\langle u, v\rangle = 0}$ ✓ ← equação.

$\langle (1,5), (7,a)\rangle = 0$

$7 + 5a = 0 \Rightarrow a = \frac{-7}{5}$

$\langle v, w\rangle = 21 + 4a = \boxed{21 - \frac{28}{5}}$

c) $\|u + v - 2w\|$

2 formas:

$$\|u + v - 2w\|^2 = \langle \underbrace{u + v - 2w}_{(1,5) + (3,4) + (7, \frac{-7}{5})}, u + v - 2w\rangle$$

$$\langle (11, \tfrac{38}{5}), (11, \tfrac{38}{5})\rangle = 121 + \frac{38^2}{25}$$

$$= \langle u, u+v-2w \rangle + \langle v, u+v-2w \rangle - 2\langle w, u+v-2w \rangle$$

$$= \langle u,u \rangle + \langle u,v \rangle - 2\langle u,w \rangle +$$
$$+ \langle v,u \rangle + \langle v,v \rangle - 2\langle v,w \rangle +$$
$$-2\langle w,u \rangle - 2\langle w,v \rangle + 4\langle w,w \rangle$$

$$\left(\begin{aligned} \langle u,u \rangle &= \langle (1,5),(1,5) \rangle = 1+25 = 26 \\ \langle v,v \rangle &= \langle (3,4),(3,4) \rangle = 9+16 = 25 \\ \langle w,w \rangle &= \langle (7, -\tfrac{7}{5}), (7, -\tfrac{7}{5}) \rangle = 49 - \tfrac{49}{25} \end{aligned} \right)$$

$$= 26 + 23 - 2(0) +$$
$$+ 23 + 25 - 2\left(21 - \frac{28}{5}\right) +$$
$$-2(0) - 2\left(21 - \frac{28}{5}\right) + 4\left(49 - \frac{49}{25}\right)$$

$$= 26 + 2(23) + 25 - 4\left(21 - \frac{28}{5}\right) + 4\left(49 - \frac{49}{25}\right)$$

$$= 97 + 4\left(49 - 21 + \frac{28}{5} - \frac{49}{25}\right)$$

$$= 97 + 4\left(28 + \frac{140 - 49}{25}\right)$$

$$= 97 + 112 + \frac{91}{25}$$

$$= 209 + \frac{91}{25}$$

$$\| u+v-2w \| = \sqrt{209 + \frac{91}{25}}$$

$P(\mathbb{R})$ $\quad \langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$

Complemento ortogonal

Obtenha o conjunto dos vetores de $\mathbb{R}^3$ ortogonais a u = (1, 3, -4)

$$u^\perp = \{ v \in \mathbb{R}^3 \mid \langle v, (1,3,-4) \rangle = 0 \}$$

Complemento ortogonal de u, é um subespaço do $\mathbb{R}^3$

$v = (x, y, z)$

$\langle (x,y,z), (1,3,-4) \rangle = 0$ ← equação.

$\{ x + 3y - 4z = 0$ ← 1 eq. 3 inc.

→ eq. de plano no $\mathbb{R}^3$
u: vetor normal.
passa pela origem.

2 variáveis livres.

Escolho $y = 1$, $z = 0$:

$x = -3y + 4z$

$x = -3$

$v_1 = (-3, 1, 0)$

Escolho $y = 0$ $z = 1$

$x = -3y + 4z$

$x = 4$

$v_2 = (4, 0, 1)$

Ou então:

$$\boxed{v = (-3y + 4z,\ y,\ z)}$$

$$= y\,(-3, 1, 0) + z\,(4, 0, 1)$$

$$y \quad v_1 \quad + \quad z \quad v_2 .$$

$$u^{\perp} = [\,(-3,1,0),\ (4,0,1)\,]$$

$$S = \{u, v\}$$

$$S^{\perp} = \{ w \in V \mid \langle u, w\rangle = 0 \ \text{e} \ \langle v, w\rangle = 0 \}$$

$$\begin{cases} \langle u, w\rangle = 0 \\ \langle v, w\rangle = 0 . \end{cases} \quad 2\ \text{eq.}$$

Bases ortogonais / ortonormais:

$$S = \{u_1, \dots, u_n\} \subset V$$

e $\langle u_i, u_j\rangle = 0$ para $i \neq j$

S é ortogonal

Se, além disso, $\langle u_i, u_i\rangle = 1 \Rightarrow S$ é

ORTONORMAL

S é ortonormal $\iff \langle u_i, u_j \rangle = \delta_{ij} = \begin{cases} 1, & \text{se } i = j \\ 0, & \text{se } i \neq j \end{cases}$

DELTA DE KRONECKER

S é L.I.

$v \in [S]$

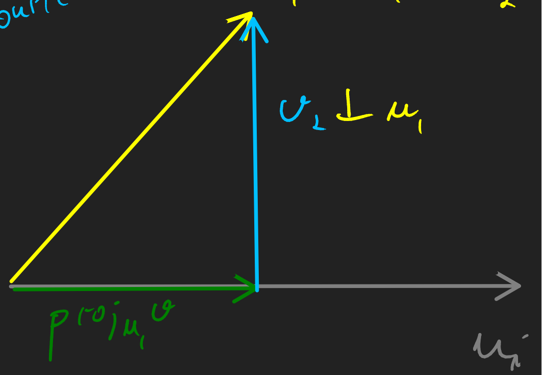
$\Updownarrow$

$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$

S ortogonal: $\Rightarrow$

$$\alpha_i = \frac{\langle u_i, v \rangle}{\langle u_i, u_i \rangle} \nearrow \text{coef. Fourier}$$

$v = proj_{u_1} v + v_2$



$$\frac{\langle u_i, v \rangle}{\langle u_i, u_i \rangle} u_i = proj_{u_i} v$$

Ex.: $\mathbb{R}^3$:

$$S = \{ \underset{v_1}{(1,2,1)}, \underset{v_2}{(2,1,-4)}, \underset{v_3}{(3,-2,1)} \}$$

1) Mostre que S é ortogonal mas não ortonormal ✓
2) Escreva as coordenadas de u=(5,3,-2) na base S

1) $v_1 \perp v_2 \Rightarrow \langle (1,2,1), (2,1,-4) \rangle = 2 + 2 - 4 = 0$ ✓

$v_1 \perp v_3 = \langle (1,2,1), (3,-2,1) \rangle = 3 - 4 + 1 = 0$ ✓

$v_2 \perp v_3 = \langle (2,1,-4), (3,-2,1) \rangle = 6 - 2 - 4 = 0$ ✓

S é ortogonal

$\|v_1\|^2 = \langle (1,2,1), (1,2,1) \rangle = 1 + 4 + 1 = 6 \neq 1$

S não é ortonormal

$\|v_2\|^2 = \langle (2,1,-4), (2,1,-4) \rangle = 4 + 1 + 16 = 21$

$\|v_3\|^2 = \langle (3,-2,1), (3,-2,1) \rangle = 9 + 4 + 1 = 14$

2) $\left[(5,3,-2)\right]_B = u$

$$(5,3,-2) = \alpha_1(1,2,1) + \alpha_2(2,1,-4) + \alpha_3(3,-2,1)$$

$$\alpha_1 = \frac{\langle v_1, u\rangle}{\langle v_1, v_1\rangle} = \frac{\langle (1,2,1),(5,3,-2)\rangle}{\langle (1,2,1),(1,2,1)\rangle} = \frac{5+6-2}{6}$$

$$= \frac{9}{6} = \frac{3}{2}$$

$$\alpha_2 = \frac{\langle v_2, u\rangle}{\langle v_2, v_2\rangle} = \frac{\langle (2,1,-4),(5,3,-2)\rangle}{21} = \frac{10+3+8}{21}$$

$$= 1$$

$$\alpha_3 = \frac{\langle v_3, u\rangle}{\langle v_3, v_3\rangle} = \frac{\langle (3,-2,1),(5,3,-2)\rangle}{14} = \frac{15-6-2}{14}$$

$$= \frac{7}{14} = \frac{1}{2}$$

$$\left[(5,3,-2)\right]_B = \begin{bmatrix} 3/2 \\ 1 \\ 1/2. \end{bmatrix}$$

$$(5,3,-2) = \frac{3}{2}(1,2,1) + (2,1,-4) + \frac{1}{2}(3,-2,1)$$

$$= \left(\frac{3}{2} + 2 + \frac{3}{2},\ 3 + 1 - 1,\ \frac{3}{2} - 4 + \frac{1}{2}\right)$$

$$= (5,\ 3,\ -2) \checkmark$$

Processo de Ortogonalização de Gram-Schmidt

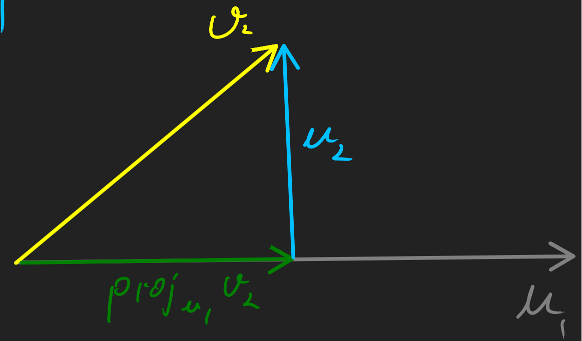
Teorema: todo espaço vetorial de dimensão finita com prod. interno possui base ortonormal

EXEMPLO $\mathbb{R}^3$ $\quad B = \{(1,0,1), (0,0,1), (0,1,1)\}$ base.

Obter uma base ortonormal a partir de B.

$C = \{u_1, u_2, u_3\}$ ortogonal

$G = \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_3\|} \right\}$



$u_1 = v_1$

$u_2 \perp u_1$

$u_2 = v_2 - proj_{u_1} v_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1$

$u_3 \perp u_2, \quad u_3 \perp u_1$

$u_3 = v_3 - proj_{u_2} v_3 - proj_{u_1} v_3 = v_3 - \frac{\langle u_2, v_3 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle u_1, v_3 \rangle}{\langle u_1, u_1 \rangle} u_1$

$B = \{(1,0,1), (0,0,1), (0,1,1)\}$

$C = \{(1,0,1), (-1,0,1), (0,1,0)\} \leftarrow$ ortogonal (verifique)

$u_1 = v_1$

$u_2 = v_2 - \frac{\langle u_1, v_2 \rangle}{\langle u_1, u_1 \rangle} u_1 = (0,0,1) - \frac{\langle (1,0,1), (0,0,1) \rangle}{\langle (1,0,1), (1,0,1) \rangle} (1,0,1)$

$u_2 = (0,0,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 0, \frac{1}{2}\right)$

Para facilitar: $u_2' = 2u_2 = (-1,0,1)$

$$u_3 = (0,1,1) - \frac{\langle (-1,0,1), (0,1,1) \rangle}{\langle (-1,0,1), (-1,0,1) \rangle}(-1,0,1) - \frac{\langle (1,0,1), (0,1,1) \rangle}{\langle (1,0,1), (1,0,1) \rangle}(1,0,1)$$

$$= (0,1,1) - \frac{1}{2}(-1,0,1) - \frac{1}{2}(1,0,1)$$

$$= (0,1,0)$$

$$G = \left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|}, \frac{u_3}{\|u_1\|} \right\} =$$

$$= \left\{ \frac{(1,0,1)}{\sqrt{2}}, \frac{(-1,0,1)}{\sqrt{2}}, (0,1,0) \right\}$$

$$G = \left\{ \frac{1}{\sqrt{2}}(1,0,1), \frac{1}{\sqrt{2}}(-1,0,1), (0,1,0) \right\} \text{ ortonormal.}$$