Revisão aulas 7-10

Lista de exercícios
Revisão 1.2 — Produto interno
Álgebra Linear

6. Seja $M_{2\times 2}(\mathbb{R})$ o espaço vetorial das matrizes 2×2 reais com o produto interno dado por

$$\langle A, B\rangle = \operatorname{Tr}(B^T A),$$

onde $\operatorname{Tr}(A)$ é o traço da matriz A e B^T é a transposta da matriz B.

(a) Obtenha uma expressão para $\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\rangle$

(b) Mostre que a base canônica desse espaço, $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, é ortonormal.

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

$$\langle A, B \rangle = tr(B^T A)$$

Se $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ então $B^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ \vdots & \vdots \end{pmatrix}$$

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad tr(C) = a + d$$

$$\langle A, B \rangle = tr(B^t A)$$

$$= tr \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha + \delta$$

(a) Obtenha uma expressão para $\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\rangle$

$$\begin{array}{cc} A & B \end{array}$$

$$B^T = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$$

$$B^T A = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} b_{11} a_{11} + b_{21} a_{21} & * \\ * & b_{12} a_{12} + b_{22} a_{22} \end{pmatrix}$$

$$\langle A, B \rangle = tr(B^t A) = b_{11} a_{11} + b_{21} a_{21} + b_{12} a_{12} + b_{22} a_{22} \quad //$$

(a) Obtenha uma expressão para $\left\langle \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right\rangle$

$$\langle A, B \rangle = tr(B^t A) = b_{11} a_{11} + b_{12} a_{12} + b_{21} a_{21} + b_{22} a_{22}$$

(b) Mostre que a base canônica desse espaço, $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, ortonormal.

$\quad v_1 \quad v_2 \quad v_3 \quad v_4$

$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{se } i = j \\ 0 & \text{se } i \neq j \end{cases}$$

$1 = \langle v_1, v_1 \rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle = 1$ ✓

$0 = \langle v_1, v_2 \rangle = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle = 1 \cdot 0 + 0 \cdot 1 + 0 + 0 = 0$ ✓

$0 = \langle v_1, v_3 \rangle$ $\quad\vdots$

$0 = \langle v_1, v_4 \rangle$

$1 = \langle v_2, v_2 \rangle$

$0 = \langle v_2, v_3 \rangle$

$0 = \langle v_2, v_4 \rangle$

$1 = \langle v_3, v_3 \rangle$

$0 = \langle v_3, v_4 \rangle$

$1 = \langle v_4, v_4 \rangle$

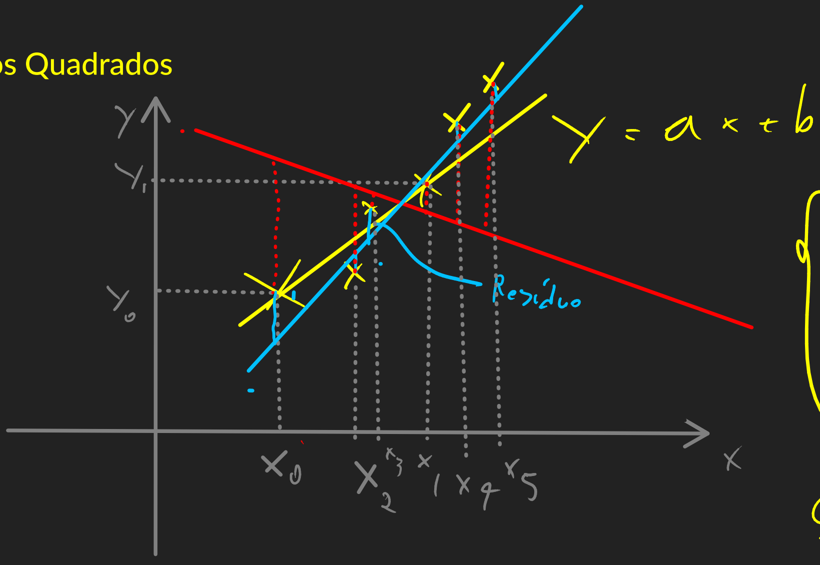
$M_{2\times 2}(\mathbb{C}) : \langle A, B \rangle = tr(B^\dagger A)$

$= \overline{b_{11}} a_{11} + \overline{b_{12}} a_{12} \ldots$

$B^\dagger = \overline{B}^t$

$\langle A, \alpha B \rangle = \overline{\alpha} \langle A, B \rangle$

Mínimos Quadrados



$$\begin{cases} a x_0 + b = y_0 \\ a x_1 + b = y_1 \end{cases}$$

Sempre tem sol. única.

Espaço vetorial das funções f(x)

$$f(x) = a x + b \cdot 1$$

modelo

$$f = a g_0 + b g_1$$

→ vetor do subespaço gerado por $\{g_1 = x, g_2 = 1\}$.

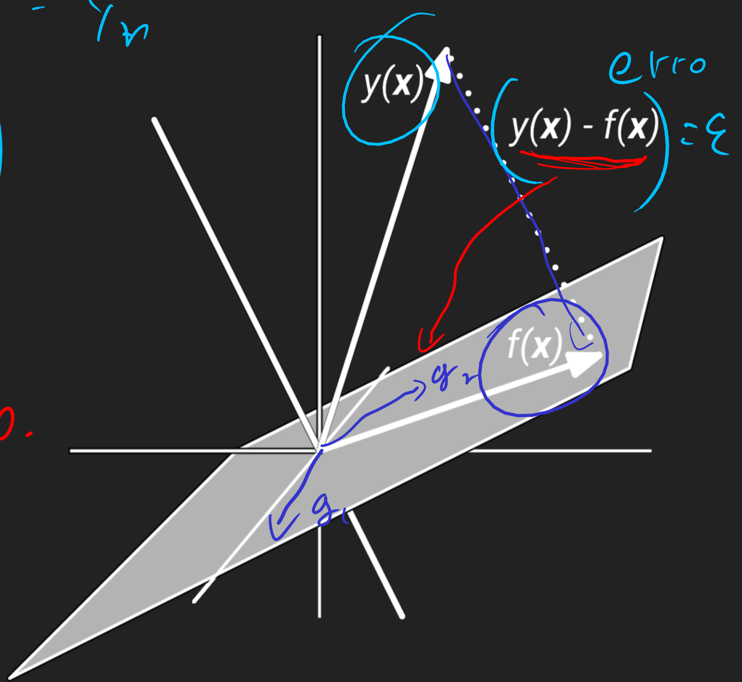
Função MEDIDA/OBSERVAÇÃO

$y(x)$

x	y	
x_1	—	$y_1 = y(x_1)$
x_2	—	$y_2 = y(x_2)$
⋮	—	⋮
x_n	—	y_n

$f(x_1)$, $g_1(x_1)$

$$y(x) = f(x) + \underbrace{\varepsilon(x)}_{\text{Mínimo.}}$$



$y(x) - f(x)$ seja ortogonal a todos os $g_i(x)$.

$$\begin{cases} \langle y-f, g_1 \rangle = 0 \\ \langle y-f, g_2 \rangle = 0 \\ \vdots \\ \langle y-f, g_n \rangle = 0 \end{cases}$$

$$\begin{cases} \langle y - \alpha_1 g_1 - \alpha_2 g_2 - \dots - \alpha_n g_n , g_1 \rangle = 0 \\ \langle y - \alpha_1 g_1 - \alpha_2 g_2 - \dots - \alpha_n g_n , g_2 \rangle = 0 \\ \vdots \\ \langle y - \alpha_1 g_1 - \alpha_2 g_2 - \dots - \alpha_n g_n , g_n \rangle = 0 \end{cases}$$

$$\begin{cases} \langle y, g_1 \rangle - \alpha_1 \langle g_1, g_1 \rangle - \alpha_2 \langle g_2, g_1 \rangle - \dots - \alpha_n \langle g_n, g_1 \rangle = 0 \\ \\ \langle y, g_n \rangle - \alpha_1 \langle g_1, g_n \rangle - \alpha_2 \langle g_2, g_n \rangle - \dots - \alpha_n \langle g_n, g_n \rangle = 0 \end{cases}$$

$$\begin{cases} \alpha_1 \langle g_1, g_1 \rangle + \alpha_2 \langle g_2, g_1 \rangle + \dots + \alpha_n \langle g_n, g_1 \rangle = \langle y, g_1 \rangle \\ \vdots \\ \alpha_1 \langle g_1, g_n \rangle + \alpha_2 \langle g_2, g_n \rangle + \dots + \alpha_n \langle g_n, g_n \rangle = \langle y, g_n \rangle \end{cases}$$

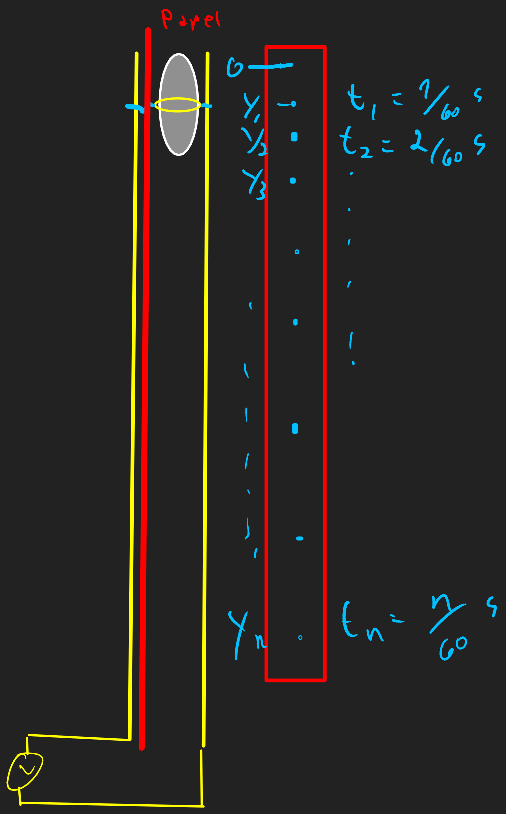
$$\begin{bmatrix} \langle g_1, g_1 \rangle & \langle g_2, g_1 \rangle & \cdots & \langle g_n, g_1 \rangle \\ \vdots & \ddots & & \vdots \\ \langle g_1, g_n \rangle & \cdots & \cdots & \langle g_n, g_n \rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \langle y, g_1 \rangle \\ \vdots \\ \langle y, g_n \rangle \end{bmatrix}$$

Definição de $\langle f, g \rangle$:

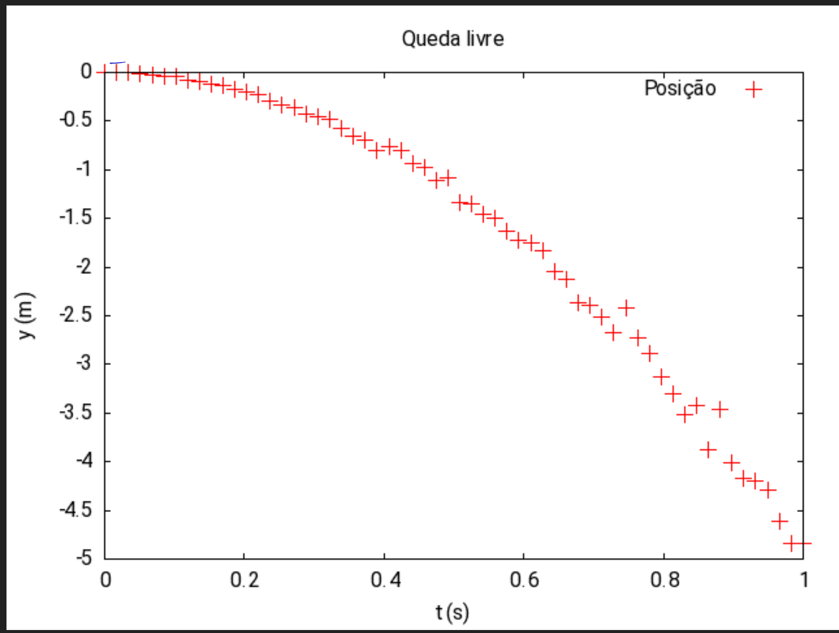
$$\langle f, g \rangle = \sum_{i=1}^{N} f(x_i)\, g(x_i)$$

$$= f(x_1)\, g(x_1) + f(x_2)\, g(x_2) + \dots$$

Exemplo: experimento da queda do ovo.



Faiscador 60Hz



Modelo: queda livre:

$$f(t) = y_0 1 + v_0 t + \frac{1}{2} a t^2$$

PARÁBOLA

Parâmetros.

Projetar $y(t)$ no espaço gerado por

$$\longrightarrow \{ g_0 = 1,\ g_1 = t,\ \boxed{g_2 = \frac{1}{2} t^2} \}$$

$f(t)$ é combinação linear dos g

Definir $\langle f, g \rangle = \sum_{i=1}^{60} f(t_i)\, g(t_i) = f(\tfrac{1}{60})\, g(\tfrac{1}{60}) + f(\tfrac{2}{60})\, g(\tfrac{2}{60})$

$$t_i = \frac{i}{60} \text{ s.}$$

$$y(\tfrac{1}{60}) = y_1$$

$\vdots$

$$f(t) = y_0 + v_0 t + a \frac{1}{2} t^2$$

$$\begin{cases} \langle y - f, g_0 \rangle = 0 \\ \langle y - f, g_1 \rangle = 0 \\ \langle y - f, g_2 \rangle = 0 \end{cases}$$

$$\begin{cases} y_0 \langle g_0, g_0 \rangle + v_0 \langle g_1, g_0 \rangle + a \langle g_2, g_0 \rangle = \langle y, g_0 \rangle \\ y_0 \langle g_0, g_1 \rangle + v_0 \langle g_1, g_1 \rangle + a \langle g_2, g_1 \rangle = \langle y, g_1 \rangle \\ y_0 \langle g_0, g_2 \rangle + v_0 \langle g_1, g_2 \rangle + a \langle g_2, g_2 \rangle = \langle y, g_2 \rangle \end{cases}$$

$$\underbrace{\begin{bmatrix} \langle g_0, g_0 \rangle & \langle g_1, g_0 \rangle & \langle g_2, g_0 \rangle \\ \langle g_0, g_1 \rangle & \langle g_1, g_1 \rangle & \langle g_2, g_1 \rangle \\ \langle g_0, g_2 \rangle & \langle g_1, g_2 \rangle & \langle g_2, g_2 \rangle \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y_0 \\ v_0 \\ a \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} \langle y, g_0 \rangle \\ \langle y, g_1 \rangle \\ \langle y, g_2 \rangle \end{bmatrix}}_{b}$$

Preciso calcular os seguintes produtos internos

$\langle g_0, g_0 \rangle = 60$ $\quad \langle g_1, g_1 \rangle = 20.16$ $\quad \langle y, g_0 \rangle =$

$\langle g_0, g_1 \rangle = 30$ $\quad \langle g_1, g_2 \rangle =$ $\quad \langle y, g_1 \rangle =$

$\langle g_0, g_2 \rangle = 10.08$ $\quad \langle g_2, g_2 \rangle =$ $\quad \langle y, g_2 \rangle =$

$$f(t) = y_0 + v_0\, t + a\,\frac{t^2}{2}$$

$$y_0 = -0{,}01026 \quad m$$

$$v_0 = 0{,}07156 \quad m/s$$

$$a = -9{,}89425 \quad m/s^2$$