Definição

Seja V um espaço vetorial sobre $\mathbb{R}$. Um $\boxed{\text{produto interno}}$ transforma cada par ordenado de vetores $(\boldsymbol{u}, \boldsymbol{v})$ de V em um único número real denotado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ e obedece aos seguintes axiomas:

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
2. $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
3. $\langle \boldsymbol{u}, \alpha\boldsymbol{v} \rangle = \alpha \langle \boldsymbol{v}, \boldsymbol{u} \rangle$
4. $\langle \boldsymbol{u}, \boldsymbol{u} \rangle \geqslant 0$, sendo que $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0 \Leftrightarrow \boldsymbol{u} = \boldsymbol{0}$

No caso complexo

Seja V um espaço vetorial sobre $\mathbb{C}$.

- O produto interno é um número complexo.
- O Axioma 1 passa a ser $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \overline{\langle \boldsymbol{v}, \boldsymbol{u} \rangle}$

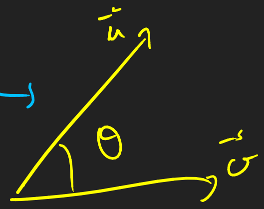
$z = a + bi \Rightarrow \bar{z} = a - bi$

Em VGA: PRODUTO ESCALAR

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

norma — dir/sentido

Norma
direção →
sentido



$u \perp v \Leftrightarrow \vec{u} \cdot \vec{v} = 0$

Na base canônica $\{\hat{i}, \hat{j}, \hat{k}\}$, $\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$ ← P.I. no $\mathbb{R}^3$ (usual)

$\vec{u} = (u_x, u_y, u_z) \in \mathbb{R}^3$

No $\mathbb{R}^n$, o produto interno $\boxed{\text{usual}}$:

$u = (u_1, \ldots, u_n) \qquad v = (v_1, \ldots, v_n)$

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

7.2.3. Exercício

1. Verifique que o produto interno usual sobre o $\mathbb{C}^n$ dado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} \overline{u_i} v_i$ satisfaz os axiomas de produto interno.

7.5. Resposta do exercício

Na aula síncrona.

7.2.3. Exercício

1. Verifique que o produto interno usual sobre o $\mathbb{C}^n$ dado por $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^{n} \overline{u_i} v_i$ satisfaz os axiomas de produto interno.

$$u = (u_1, u_2, \ldots, u_n) \in \mathbb{C}^n$$

$$u_i \in \mathbb{C}$$

$$v = (v_1, \ldots, v_n) \in \mathbb{C}^n$$

$$v_i \in \mathbb{C}$$

1) $\langle v, u \rangle = \overline{\langle u, v \rangle}$ Note $\overline{\langle u, v \rangle} = \overline{\left(\sum_i^n \overline{u_i} v_i \right)}$

Calculando $\langle v, u \rangle$:

Por def.

$$\langle v, u \rangle = \sum_{i=1}^{n} \overline{v_i} u_i = \overline{v_1} u_1 + \overline{v_2} u_2 + \cdots + \overline{v_n} u_n$$

$$= \sum_{i=1}^{n} u_i \overline{v_i}$$

Note: $\overline{(\overline{z})} = z$

$$= \sum_{i=1}^{n} \overline{\overline{u_i}}\, \overline{v_i}$$

$\left(\overline{a} \cdot \overline{b} = \overline{ab} \right)$ I

$$= \sum_{i=1}^{n} \overline{(\overline{u_i} v_i)}$$

$\left(\overline{a} + \overline{b} = \overline{(a+b)} \right)$

Exercício

$$= \overline{\left(\sum_{i=1}^{n} \overline{u_i} v_i \right)}$$

$$= \overline{\langle u, v \rangle}$$

I $(a - bi)(c - di) =$
$= ac - bci - adi - bd$
$= \overline{(ac - bd) + (bc + ad) i}$

2) $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

$$\langle u, v+w \rangle = \sum_{i=1}^{n} \bar{u}_i (v_i + w_i)$$

$$= \sum_i [\bar{u}_i v_i + \bar{u}_i w_i]$$

$(\bar{u}_1 v_1 + u_1 w_1) + (\bar{u}_2 v_2 + \bar{u}_2 w_2) + \dots$

$$= \sum_i \bar{u}_i v_i + \sum_i \bar{u}_i w_i$$

$$= \langle u, v \rangle + \langle u, w \rangle \checkmark$$

3) $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$

$$\langle u, \alpha v \rangle = \sum_i \bar{u}_i (\alpha v_i)$$

$\bar{u}_1 (\alpha v_1) + \bar{u}_2 (\alpha v_2) + \dots$

$\alpha (\bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots)$

$$= \alpha \sum_i \bar{u}_i v_i$$

$$= \alpha \langle u, v \rangle.$$

4) $\langle u, u \rangle \geq 0 \quad (\langle u, u \rangle \in \mathbb{R})$

$\langle u, u \rangle = 0 \iff u = 0$

$a, b \in \mathbb{R}$

$\bar{z} z = (a+bi)(a-bi)$

$= a^2 - b^2 i^2$

$= \boxed{a^2 + b^2} \in \mathbb{R} \geq 0$

$$\langle u, u \rangle = \sum_i \bar{u}_i u_i$$

$$= \sum_i \underbrace{(\mathrm{Re}(u_i))^2}_{\geq 0} + \underbrace{(\mathrm{Im}(u_i))^2}_{\geq 0}$$

$$\geq 0 \quad \forall u \in \mathbb{C}^n$$

$$\sum_i \bar{u}_i u_i = 0 \iff \mathrm{Re}(u_i) = \mathrm{Im}(u_i) = 0 \quad \forall i$$

$$u = 0$$

Logo essa operação é prod. interno em $\mathbb{C}^n$.

No $\mathbb{R}^n$: Na base canônica $C = \{(1,0,\dots,0),(0,1,0,\dots,0), \dots\}$

$$[u]_C = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad [v]_C = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Prod. Int. usual: $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

$$\langle u, v \rangle = [u_1 \ u_2 \ \cdots \ u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = [u]^T [v]$$

No $\mathbb{C}^n$: prod. int. usual

$$\langle u, v \rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \dots + \bar{u}_n v_n$$

Base canônica:

$$C = \{(1+0i, 0, \dots, 0), (0, 1+0i, 0, \dots, 0), \dots, (0, \dots, 1+0i)\}$$

$$(1, 0, \dots, 0) \quad \dots \quad (0, 0, \dots, 1)$$

$$u = (u_1, \ldots, u_n) \qquad [u] = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$v = (v_1, \ldots, v_n) \qquad [v] = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\langle u, v \rangle = \begin{bmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \overline{[u]^T} [v]$$

$$= [u]^\dagger [v]$$

$[u]^\dagger$ matriz hermiteana conjugada de [u]

Contraexemplo:

$\mathbb{R}^2$:

$\langle u, v\rangle = u_1 v_1 u_2 v_2$ $\boxed{\text{Não}}$ é prod. interno.

1) $\langle u, v\rangle = \langle v, u\rangle$ OK

3) $\langle u, \alpha v\rangle = u_1(\alpha v_1)\, u_2(\alpha v_2) = \alpha^2 u_1 v_1 u_2 v_2$

$= \alpha^2 \langle u, v\rangle$

$\neq \alpha \langle u, v\rangle$

$\mathbb{R}^2$:

$\langle u, v\rangle = u_1 v_1 - u_2 v_1 - u_1 v_2 + 5 u_2 v_2$

1) $\langle u, v\rangle = \langle v, u\rangle$

2) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$ exercício.

3) $\langle u, \alpha v\rangle = \alpha \langle u, v\rangle$

$\langle u, \alpha v\rangle = u_1(\alpha v_1) - u_2(\alpha v_1) - u_1(\alpha v_2) + 5 u_2(\alpha v_2)$

$= \alpha \langle u, v\rangle$

4) $\langle u, u\rangle \geq 0$:

$\langle u, u\rangle = u_1 u_1 - u_1 u_2 - u_2 u_1 + 5 u_2 u_2$

$= u_1^2 - 2 u_1 u_2 + 5 u_2^2$

$= \underbrace{(u_1 - u_2)^2}_{\geq 0} + \underbrace{4 u_2^2}_{\geq 0} \geq 0$.

Seja $u = (2,-1)$ $v = (1,2)$

calcule $\langle u, v \rangle$ definido acima.

$$\langle u, v \rangle = 2\cdot 1 - 2\cdot 2 + 1\cdot 2 + 5(-1\cdot 2)$$
$$= 2 - 4 + 2 - 10 = -10$$ //

Matrizes: $M_{2\times 2}$: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}$.

$$\langle A, B \rangle = tr(B^T A)$$

I) Mostrar que é um prod. interno:

1) $\langle A, B \rangle = \langle B, A \rangle$.

$$\langle B, A \rangle = tr(A^T B)$$
$$= tr((B^T A)^T)$$
$$= tr(B^T A)$$
$$= \langle A, B \rangle.$$

$(AB)^T = B^T A^T$

$(A^T B)^T = B^T (A^T)^T$

$= B^T A$

$tr(B^T) = tr(B)$

⋮ 2-4 (Exercício).

Exemplo:

$$A = \begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & -1 \\ -2 & -1 \end{pmatrix}$$

$$\langle A, B \rangle = tr(B^t A)$$

$$= tr\left[\begin{pmatrix} 5 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 1 & 0 \end{pmatrix}\right]$$

$$= tr\left[\begin{pmatrix} 8 & * \\ * & 3 \end{pmatrix}\right] = 8+3 = 11$$

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$$

$$\langle A, B \rangle = tr(B^T A)$$

$$= tr\left[\begin{pmatrix} b_1 & b_3 \\ b_2 & b_4 \end{pmatrix}\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\right].$$

$$= tr \begin{pmatrix} b_1 a_1 + b_3 a_3 & * \\ * & b_2 a_2 + b_4 a_4 \end{pmatrix}$$

$$= b_1 a_1 + b_3 a_3 \quad + \quad b_2 a_2 + b_4 a_4$$

$$= b_1 a_1 + b_2 a_2 + b_3 a_3 + b_4 a_4$$

$$= [B]^T [A]$$ na base canônica.

Teorema. Dado esp. vetorial e prod. interno

Existe base B tal que

$$\langle u, v \rangle = [u]_B^T [v]$$

$$\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle v, w \rangle$$

Seja $B = \{v_1, \dots, v_n\}$ base de V.

$u = \alpha_1 v_1 + \cdots + \alpha_n v_n \Rightarrow [u]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$

$w = \beta_1 v_1 + \cdots + \beta_n v_n \Rightarrow [w]_B = \begin{bmatrix} \beta \\ \vdots \\ \end{bmatrix}$

$$\langle u, w \rangle = \left\langle \sum_i \alpha_i v_i , \sum_j \beta_j v_j \right\rangle$$

$$= \sum_i \sum_j \alpha_i \beta_j \underbrace{\langle v_i, v_j \rangle}$$

depende APENAS dos vetores da base.

$$= \alpha_1 \beta_1 \langle v_1, v_1 \rangle + \alpha_1 \beta_2 \langle v_1, v_2 \rangle +$$

$$+ \dots$$

Se $\langle v_i, v_i \rangle = 1$

$\boxed{\langle v_i, v_j \rangle = 0}$ se $i \neq j$ $\Big\}$ BASE ORTONORMAL.

$\langle u, v \rangle$, $\langle u, w \rangle$ $\Rightarrow$ $\langle u, 2v + 5w \rangle$