Lista 2.2

2. Considere a transformação linear $F : \mathbb{R}^3 \to P_2(\mathbb{R})$, que leva um vetor do $\mathbb{R}^3$ a um polinômio de grau $\leqslant 2$. A transformação é dada pela expressão

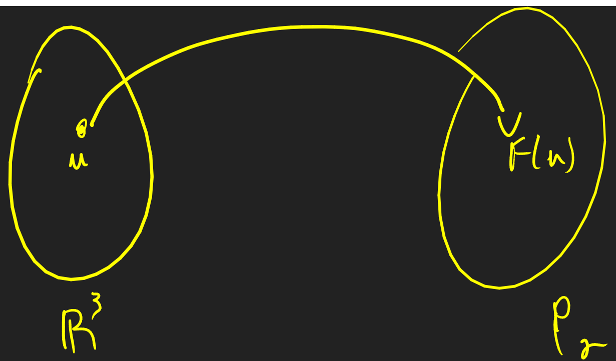
$$F(x, y, z) = \boxed{x + yt + zt^2}$$

Sejam $\boxed{B = \{1, t, t^2\}}$ a base canônica de $P_2(\mathbb{R})$ e $C = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ a base canônica de $\mathbb{R}^3$.

(a) Calcule $F(1, 0, 0)$, $F(0, 1, 0)$ e $F(0, 0, 1)$, ou seja, F aplicada à base canônica de $\mathbb{R}^3$.

(b) Obtenha as coordenadas de $F(1, 0, 0)$, $F(0, 1, 0)$ e $F(0, 0, 1)$ em relação à base B.

(c) Escreva a matriz $[F]_B^C$ em relação às bases canônicas dos espaços.



espaços vetoriais sobre os $\mathbb{R}$

$F : \mathbb{R}^3 \to P_2$

$\to F(x, y, z) = x + yt + zt^2$

a) $\boxed{F(1,0,0)} = 1 + 0t + 0t^2 = \boxed{1} \in P_2$

$\to \boxed{F(0,1,0)} = 0 + 1t + 0t^2 = \boxed{t \in P_2}$

$\to F(0,0,1) = 0 + 0t + 1t^2 = t^2 \in P$

$\hookrightarrow$ vetor de P_2

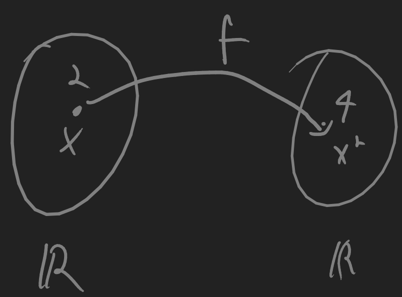
//

função real:

$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^2$

$f(2) = 4 \in \mathbb{R}$



b) obtenha das coordernadas de F(1, 0, 0), etc. em relação à base B, base canônica de P_2

$$[F(1,0,0)]_B$$

$$[F(0,1,0)]_B$$

$$[F(0,0,1)]_B$$

Anatomia da notação

$$[F(\vec{v})]_B$$

vetor de $\mathbb{R}^3$

vetor de P_2

$$[w]_B$$

matriz de coordenadas
em realação à base B

$$B = \{1, t, t^2\}$$

$$[F(1,0,0)]_B = [1]_B = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$[F(e_1)]_B$$

$$1 = 1(1) + 0t + 0t^2$$

$$[F(0,1,0)]_B = [t]_B = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$[F(e_2)]_B$$

$$[F(0,0,1)]_B = [t^2]_B = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

$$[F(e_3)]_B$$

$$[T]_B^C = \Big[\, [F(e_1)]_B \; [F(e_2)]_B \; [F(e_3)]_B \,\Big]$$

$$[T]_B^C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Escreva $[\boldsymbol{v}]_C$, a matriz de coordenadas de $\boldsymbol{v} = (2, -1, 3)$ em relação à base C.

(e) Calcule $[F(\boldsymbol{v})]_B = \boxed{[F]_B^C\,[\boldsymbol{v}]_C}$.

(f) Escreva explicitamente o polinômio $F(\boldsymbol{v})$. $\leftarrow$

(g) Calcule $F\left(x_0, v_0, \frac{g}{2}\right)$.

d) $[v]_C$ com $v = (2, -1, 3)$

$$[v]_C = [(2,-1,3)]_C = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_C$$

e) $$\underbrace{[F(v)]_B}_{\text{matriz de coordenadas}} = [F]_B^C\,[v]_C$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_C$$

$$[F(v)]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_B$$

f) $$\underbrace{F(v)}_{\text{polinômio}} = F(2,-1,3)$$

$$F(x, y, z) = x + yt + zt^2$$

$$F(2,-1,3) = 2 + (-1)t + 3t^2$$

Como $[F(v)]_B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}_B$

$B = \{v_0, v_1, v_2\} \leftarrow$

$B = \{1, t, t^2\}$

$$F(v) = 2v_0 - 1v_2 + 3v_3$$

$$F(v) = 2(1) - 1t + 3t^2 = 2 - 1t + 3t^2 \quad //$$

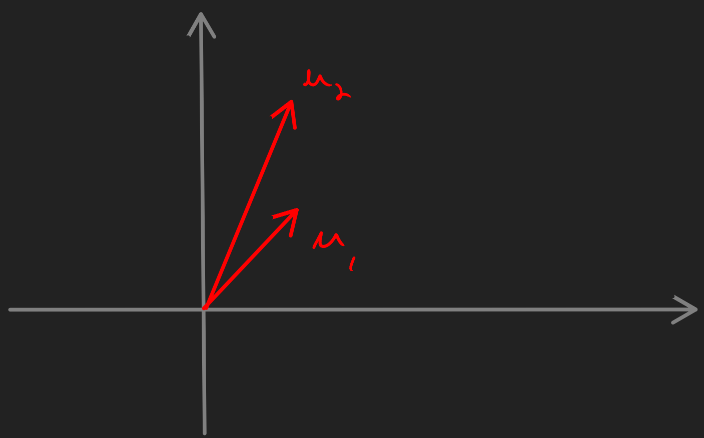
Exemplo:

$$S, T \in L(\mathbb{R}^2)$$ operadores lineares do $\mathbb{R}^2$

$$S: \mathbb{R}^2 \to \mathbb{R}^2 \quad S(x,y) = (x+2y, 4x)$$

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(x,y) = (y, x+3y)$$

$$B = \{ u_1 = (1,1), u_2 = (1,2) \}$$ base de $\mathbb{R}^2$



a) obtenha [S], a matriz de S na base B

$$[S]^B_B = [S]_B = [S]$$

$$[S] = \left[\overbrace{[\underbrace{S(u_1)}_{\text{vetor}}]_B}^{\text{matriz coluna}} \quad [S(u_2)]_B \right]$$

$$= \left[[S(1,1)]_B \quad [S(1,2)]_B \right]$$

$$S(1,1) = (1+2(1), 4(1)) = (3,4)$$ $$S(x,y) = (x+2y, 4x)$$

$$S(1,2) = (1+2(2), 4(1)) = (5,4)$$

Coordenadas de (x,y) na base B.

$$[(x,y)]_B = \begin{bmatrix} a \\ b \end{bmatrix}_B \quad \text{tal que} \quad (x,y) = a u_1 + b u_2$$

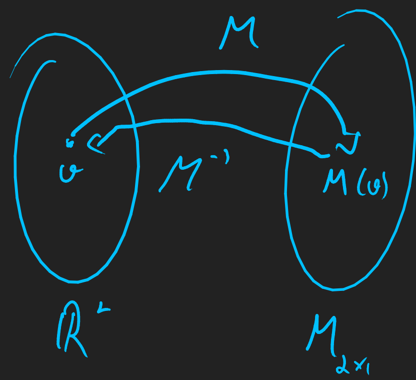
$$(x,y) = a(1,1) + b(1,2)$$ equação vetorial

$$= (a+b,\ a+2b)$$

$$\begin{cases} a+b = x \\ a+2b = y \end{cases} \Rightarrow \begin{cases} a = x-b \\ x-b+2b = y \end{cases} \Rightarrow \left\{ b = y-x \right.$$

$$a = x-(y-x) = 2x-y$$

$$\rightarrow \boxed{[(x,y)]_B = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix}_B}$$



$$\begin{cases} M_B : \mathbb{R}^2 \to M_{2\times 1} \\ M_B(x,y) = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix} \\ M^{-1}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a(1,1) + b(1,2) \\ \qquad = (a+b,\ a+2b) \end{cases}$$

transformação linear bijetora (isomorfismo)

$$[(x,y)]_B = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix}_B$$

$$S(1,1) = (1+2(1), 4(1)) = (3,4)$$

$$S(1,2) = (1+2(2), 4(1)) = (5,4)$$

$$[\underbrace{S(1,1)}_{\text{vetor}}] = [\underbrace{(3,4)}_{\text{vetor}}] = \begin{bmatrix} 2(3)-4 \\ -3+4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_B$$

$$[\underbrace{S(1,2)}_{\text{vetor}}] = [\underbrace{(5,4)}_{\text{vetor}}] = \begin{bmatrix} 2(5)-4 \\ -5+4 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}_B$$

$$[S] = \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} //$$

b) Seja $v = (15, -10)$

Calcule [S(v)], a matriz de coordenadas de S(v) na base B

$$S(x,y) = (x+2y, 4x)$$

2 abordagens

$$[S(v)] = [\underbrace{S(15,-10)}] = [(15+2(-10), 4(15))]$$

$$= [(-5, 60)] =$$

$$[(x,y)]_B = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix}_B$$

$$= \begin{bmatrix} 2(-5)-60 \\ -(-5)+60 \end{bmatrix} = \begin{bmatrix} -70 \\ 65 \end{bmatrix} //$$

Alternativamente

$$[(x,y)]_B = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix}_B$$

$$[S(v)] = [S][v]$$

$$[v] = [(15,-10)] = \begin{bmatrix} 2(15)-(-10) \\ -15+(-10) \end{bmatrix} = \begin{bmatrix} 40 \\ -25 \end{bmatrix}$$

$$[S(v)] = \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 40 \\ -25 \end{bmatrix} = \begin{bmatrix} 2(40)+6(-25) \\ 40-(-25) \end{bmatrix}$$

$$= \begin{bmatrix} 80-150 \\ 40+25 \end{bmatrix} = \begin{bmatrix} -70 \\ 65 \end{bmatrix} //$$

c) Obtenha [T] $\quad T(x,y) = (y, x+3y)$

$$[T] = \Big[[T(u_1)] \quad [T(u_2)]_B \Big]$$

$$T(1,1) = (1, 1+3(1)) = (1,4)$$

$$T(1,2) = (2, 1+3(2)) = (2,7)$$

$$[(x,y)]_B = \begin{bmatrix} 2x-y \\ -x+y \end{bmatrix}_B$$

$$[T(1,1)] = [(1,4)] = \begin{bmatrix} 2(1)-4 \\ -1+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$[T(1,2)] = [(2,7)] = \begin{bmatrix} 2(2)-7 \\ -2+7 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$

$$[T] = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} //$$

$$[S] = \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} \qquad [T] = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$$

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d) $[S+T]$

Adição de transf.

$$(S + T)(v) = S(v) + T(v)$$

adição de transformações

Adição no $\mathbb{R}^2$

$$S, T \in \underbrace{L(\mathbb{R}^2)}$$

espaço vetorial dos operadores lineares de $\mathbb{R}^2$

$$M_B : L(\mathbb{R}^2) \to M_{2\times 2} \to \text{transf. linear.}$$

$$M(S) = \underbrace{\begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix}}_{\text{matriz}}$$

op. linear

$$M(S+T) = M(S) + M(T)$$

d) $[S+T] = M_B(S+T) =$

$$= M_B(S) + M_B(T)$$

$$= [S] + [T]$$

adição de matrixes 2×2

$$[S+T] = [S] + [T]$$

$$= \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 6-3 \\ 1+3 & -1+5 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 4 & 4 \end{bmatrix}$$

Alternativamente:

$$[S+T] = \begin{bmatrix} [(S+T)(u_1)] & [(S+T)(u_2)] \end{bmatrix}$$

$$(S+T)(u_1) = S(u_1) + T(u_1)$$

$$= S(1,1) + T(1,1)$$

$$= (3,4) + (1,4) = (4,8) = 4u_2$$

$$[(S+T)(u_1)] = [(4,8)] = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

Analogamente

$$(S+T)(u_2) = S(u_2) + T(u_2)$$

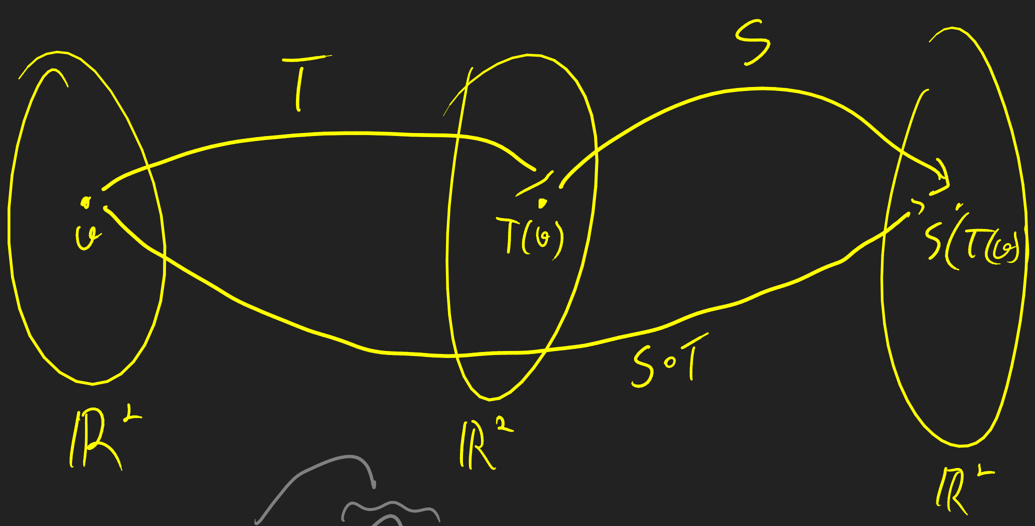
$$= (5,4) + (2,7) = (7,11)$$

$$[(S+T)(u_2)] = [(7,11)] = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$[S+T] = \begin{bmatrix} 0 & 3 \\ 4 & 4 \end{bmatrix}$$

$$\begin{aligned}
[(S+T)(v)] &= [S(v) + T(v)] \\
&= [S(v)] + [T(v)] \\
&= [S][v] + [T][v] \\
&= \big(\underbrace{[S]+[T]}\big)[v] \\
&= [S+T] \quad [v]
\end{aligned}$$

Composição:



$$(S \circ T)(v) = S(T(v))$$

e) Calcule [S ○ T], a matriz da transformação S composta com T

$$[S\circ T] = \begin{bmatrix} [(S\circ T)(u_1)] & [(S\circ T)(u_2)] \end{bmatrix}$$

$$[(S\circ T)(u_1)]$$

2 formas:

$$-\quad [(S\circ T)(u_1)] = [\,S(\overbrace{T(u_1)}^{(x,y)})\,] \Rightarrow S(T(u_1)) = a u_1 + b u_2 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix}$$

$$S(T(1,1)) = S(1, 1+3)$$

ou:

$$[(S\circ T)(u_1)] = [S(T(u_1))]$$

$$[S(v)] = [S][v]$$

Se $v = T(u_1)$

$$[v] = [T(u_1)] = [T][u_1]$$

Logo

$$[S(T(u_1))] = [S][T(u_1)]$$
$$= [S]\left([T][u_1]\right)$$

Por construção: $[u_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $[u_2]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$[S(T(u_1))] = [S][T][u_1]$$

$$= \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-2)+6(3) & 2(-3)+6(5) \\ 1(-2)-1(3) & 1(-3)-1(5) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$[(S\circ T)(u_1)] = \begin{bmatrix} 14 & 24 \\ -5 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 14 \\ -5 \end{bmatrix}$$

$$[(S\circ T)(u_2)] = [S][T][u_2]$$

$$= \begin{bmatrix} 2 & 6 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

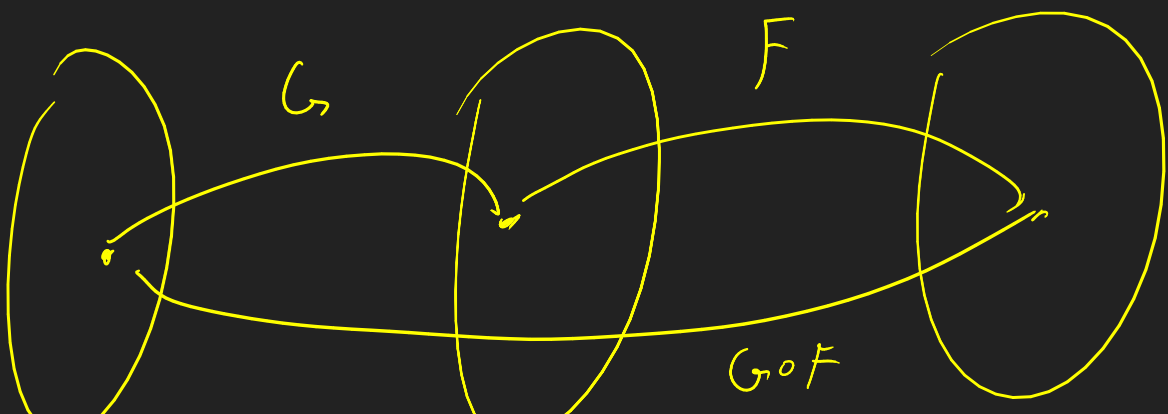
$$= \begin{bmatrix} 14 & 24 \\ -5 & -8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ -8 \end{bmatrix}$$

$$[S\circ T] = \begin{bmatrix} 14 & 24 \\ -5 & -8 \end{bmatrix} \quad //$$

$$[S\circ T] = [S][T]$$

$$[(S\circ T)(v)] = [S(T(v))]$$
$$= [S][T(v)]$$
$$= [S]([T][v]) \quad \leftarrow \text{assoc.}$$
$$= \underbrace{([S][T])}[v]$$
$$[(S\circ T)(v)] = [S\circ T][v]$$

matriz da soma de transf. = soma das matrizes de transf.
matriz do prod. de transf. por escalar = prod. da matriz de transt. por esalar
matriz da composição = produto das matrizes das transf.



$$\mathrm{Dom}(F) = CD(G)$$