Notas de aula ECT2202 T05 2021-04-15 Aulas 20 e 21 — Diagonalização

Considere o operador $F: \mathbb{R}^3 \to \mathbb{R}^3$ tal que $[F] = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ em relação à base canônica. A base em que sua matriz é diagonal é:

$B = \{v_1, v_2, v_3\}$

tal que $[F]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Isso ocorre quando v1, v2, e v3 são autovetores de F com autovalores λ1, λ2, λ3

Se
$$F(v_1) = \lambda_1 v_1 = \lambda_1 v_1 + 0 v_2 + 0 v_3$$
$$F(v_2) = \lambda_2 v_2 = 0 v_1 + \lambda_2 v_2 + 0 v_3$$
$$F(v_3) = \lambda_3 v_3 = 0 v_1 + 0 v_2 + \lambda_3 v_3$$

$$[F(v_1)]_B = [\lambda_1 v_1]_B = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} \leftarrow$$

$$[F(v_2)]_B = [\lambda_2 v_2] = \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix} \leftarrow$$

Logo $[F]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

Considere o operador $F: \mathbb{R}^3 \to \mathbb{R}^3$ tal que $[F] = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ em relação à base canônica. A base em que sua matriz é diagonal é:

Quais dos conjuntos abaixo é formado de autovetores de F?

1) Autovalores: raízes do polinômo característico

$$p_F(\lambda) = \det \begin{vmatrix} 2-\lambda & 0 & 4 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} + 4\begin{vmatrix} 1 & -1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (2-\lambda)(-1-\lambda)(2-\lambda) + 4(1+\lambda)$$

$$= (2-\lambda)^2(-1-\lambda) + 4(1+\lambda)$$

$$= (2-\lambda)^2(-1-\lambda) - 4(-1-\lambda)$$

$$= (-1-\lambda)\left[(2-\lambda)^2 - 4\right]$$

Raizes: $(-1-\lambda) = 0 \Rightarrow \boxed{\lambda = -1}$

ou $(2-\lambda)^2 - 4 = 0$

$$\lambda^2 - 4\lambda + 4 - 4 = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0 \Rightarrow \begin{cases} \lambda = 0 \\ \text{ou} \\ \lambda = 4. \end{cases}$$

Autovalores:

$$\begin{array}{ccc} \lambda_1 = 4, & \lambda_2 = 0, & \lambda_3 = -1. \\ \uparrow & \uparrow & \uparrow \\ V(\lambda_1) & V(\lambda_2) & V(\lambda_3) \\ v_1 & v_2 & v_3 \end{array} \Rightarrow \dim(\mathbb{R}^3) = 3$$

$: L.I.$

3 vetores L.I. esp. dim 3. BASE.

Autovetores:

1) $\lambda_1 = 4$ $\quad [v_1] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad F(v_1) = \lambda_1 v_1$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4 \begin{bmatrix} x \\ y \\ z \end{bmatrix} \leftarrow$$

$$\begin{cases} 2x + 4z = 4x \\ x - y = 4y \\ x + 2z = 4z \end{cases}$$

$$\begin{cases} 2x - 4x + 4z = 0 \\ x - y - 4y = 0 \\ x + 2z - 4z = 0 \end{cases}$$

$$\begin{cases} -2x + 4z = 0 \\ x - 5y = 0 \Rightarrow 2z = 5y \\ x - 2z = 0 \Rightarrow x = 2z \end{cases}$$

$$v_1 = z\left(2, \frac{2}{5}, 1\right) \Rightarrow \alpha(10, 2, 5)$$

z ← var. livre.

$V(\lambda_1) = [(10, 2, 5)]$ (escolhi $z = 5$)

$\lambda_2 = 0$.

$T(v_2) = \lambda_2 v_2 \Rightarrow T(v_2) = 0\, v_2$

$T(v_2) = \vec{0}$.

$(\ker(T))$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} 2x + 4z = 0 \\ x - y = 0 \Rightarrow x = y \\ x + 2z = 0 \Rightarrow x = -2z \end{cases}$$

$v_2 = z(-2, -2, 1)$

$V(\lambda_2) = [(-2, -2, 1)] \leftarrow$

$\lambda_3 = -1$ $\quad T(v_3) = \lambda_3 v_3$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} 2x + 4z = -x \\ x - y = -y \\ x + 2z = -z \end{cases}$$

$$\begin{cases} 3x + 4z = 0 \\ x = 0 \\ x + 3z = 0 \end{cases} \Rightarrow \begin{array}{l} x = 0 \\ 3z = x = 0 \\ y \text{ é livre} \end{array}$$

$$v_3 = y(0,1,0)$$

$$V(\lambda_3) = [(0,1,0)].$$

$$\left.\begin{array}{l} v_1 = (10,2,5) \\ v_2 = (-2,-2,1) \\ v_3 = (0,1,0) \end{array}\right\} \text{L.I.}$$

$$B = \{(10,2,5), (-2,-2,1), (0,1,0)\}$$

é base de $\mathbb{R}^3$.

$$\text{e } [F]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

○ ✓ d) $\{(10,2,5), (0,1,0), (-2,-2,1)\} = M$

$(2,2,-1)$

Na base M

$$[F]_M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = P D P^{-1}$$

↳ matriz mudança de base

A e D são semelhantes.

$C = \{(1,0,0), (0,1,0), (0,0,1)\}$

$B = \{(10,2,5), (-2,-2,1), (0,1,0)\}$.

$$\to [F(v)]_C = [F]_C [v]_C$$

$$[F(v)]_B = [F]_B [v]_B$$

Mas:

$$\hookrightarrow [F(v)]_B = [I]_B^C [F(v)]_C$$

e

$$\begin{cases} [v]_B = [I]_B^C [v]_C \\ [v]_C = \underline{[I]_C^B [v]_B} \leftarrow \end{cases}$$

$$[F(v)]_B = [I]_B^C [F(v)]_C$$

$$= [I]_B^C \left([F]_C [v]_C \right)$$

$$[F(v)]_B = \underbrace{\left([I]_B^C [F]_C [I]_C^B \right)}_{[F]_B} [v]_B$$

$$[F(v)]_B = [F]_B [v]_B$$

Logo

$$[F]_B = [I]_B^C [F]_C [I]_C^B$$

Mas $[I]_B^C = \left([I]_C^B \right)^{-1}$

Chamando $P = [I]_C^B$

$P^{-1} = [I]_B^C \quad \left[[(1,0,0)]_B \; [(0,1,0)]_B \; [(0,0,1)]_B \right]$

Logo

diag.

$$[F]_B = P^{-1} [F]_C P \qquad \text{ou}$$

A e D são semelhantes!

$$\boxed{P [F]_B P^{-1} = [F]_C} \Rightarrow \boxed{P D P^{-1} = A.}$$

diagon

diagonal

$$P: [T]^B_C = \begin{bmatrix} 10 & -2 & 0 \\ 2 & -2 & 1 \\ 5 & 1 & 0 \end{bmatrix} : \{ [v_1] \; [v_2] \; [v_3] \}$$

$$P = \left[[v_1]_C \; [v_2]_C \; [v_3]_C \right].$$

$T: \mathbb{R}^2 \to \mathbb{R}^2$

Sendo que $\begin{cases} T(1,1) = (1,1) \\ T(1,-1) = (-1,1) = -1(1,-1) \end{cases}$

Ache a matriz de T na base canônica.

$$(1,0) = \frac{1}{2}\left[(1,1) + (1,-1)\right] \qquad \begin{cases} (1,0) = a(1,1) + b(1,-1) \\ a = \frac{1}{2} \quad b = \frac{1}{2} \end{cases}$$

$$(0,1) = \frac{1}{2}\left[(1,1) - (1,-1)\right] \qquad \begin{cases} (0,1) = a(1,1) + b(1,-1) \\ a = \frac{1}{2} \quad b = -\frac{1}{2} \end{cases}$$

$$T(1,0) = \frac{1}{2}\left[T(1,1) + T(1,-1)\right] = \frac{1}{2}\left[(1,1) + (-1,1)\right]$$

$$T(0,1) = \frac{1}{2}\left[T(1,1) - T(1,-1)\right] = \frac{1}{2}\left[(1,1) - (-1,1)\right]$$

$$= \begin{matrix} T(1,0) = (0,1) \\ T(0,1) = (1,0) \end{matrix} \Rightarrow [T]_C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

T tiene autovalores

$$\begin{cases} \lambda_1 = 1 & \text{con autovector } (1,1) \\ \lambda_2 = -1 & \text{con autovector } (1,-1) \end{cases}$$

Obtener $[T]$ en base canónica