Notas de aula ECT2202 T05 2021-04-27 Aula 24 — Cônicas e quádricas

Identifique a curva abaixo e a direção de seus eixos principais.

$$\underbrace{\overset{A}{4x^2} + \overset{C}{4y^2} - \overset{B}{2xy}}_{Q \text{ (rotação) simétrica}} \overbrace{\underbrace{- 8x - 2y}_{L}}^{\text{translação}} + 3 = 0$$

$\lambda_1 (\bar{x} - \bar{x}_0)^2 + \lambda_2 (\bar{y} - \bar{y}_0)^2 + d = 0$

Equação quadrática no $\mathbb{R}^2$

λ
- Elipse (ou ponto, degenerado)
- Hipérbole (ou retas concorrentes)
- Parábola (ou retas paralelas)

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} D & E \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + F = 0$$

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{Q} + \underbrace{\begin{bmatrix} -8 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{L.} + 3 = 0$$

Para fazer a rotação dos eixos, diagonalizar a forma quadrática:

Diagonalizar $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$

1) Autovalores: poli característico:

$$p_Q(\lambda) = \det \begin{bmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 8\lambda + 16 - 1 = 0$$

$$\lambda^2 - 8\lambda + 15 = 0 \Rightarrow \begin{matrix} \lambda_1 + \lambda_2 = 8 \\ \lambda_1 \lambda_2 = 15 \end{matrix} \Rightarrow \begin{matrix} \lambda_1 = 5 > 0 \\ \lambda_2 = 3 > 0 \end{matrix}$$

$\rightarrow$ Elipse, $\boxed{\text{ponto ou conj. vazio}}$

Direção dos eixos principais: autovetores!

$\lambda_1 = 5$:

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$$

$\hookrightarrow$ simétrica → matriz ortonormal

$M^{-1} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} M = D$

M^t base ortonormal em diag.

$$\begin{cases} 4x - y = 5x \\ -x + 4y = 5y \end{cases} \Rightarrow \begin{cases} -x - y = 0 \Rightarrow x = -y \\ -x - y = 0 \end{cases}$$

$u_1 = (1, -1)$

$v_1 = x(1, -1)$

$\boxed{\text{Normalizando}}$: $v_1 = \frac{1}{\sqrt{2}}(1, -1)$ //

$\cancel{v_1} = \frac{u_1}{\|u_1\|}$ divisão.

$(1, -1) \perp v_1$

$\|u_1\| = \sqrt{\langle u_1, u_1 \rangle}$
$= \sqrt{1 \cdot 1 + (-1)(-1)}$
$= \sqrt{1+1} = \sqrt{2}$

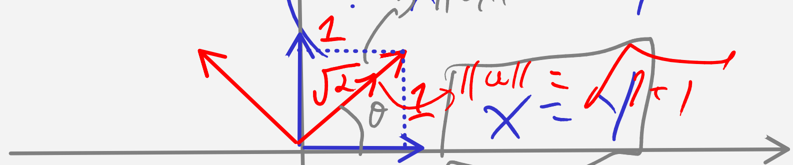
$\lambda_2 = 3$

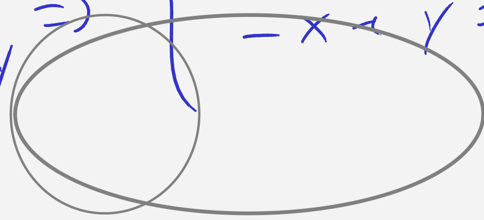
$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} 4x - y = 3x \\ -x + 4y = 3y \end{cases} \Rightarrow \begin{cases} \boxed{x - y = 0} \\ -x + y = 0 \end{cases}$$

$\|v_1\| = 1$

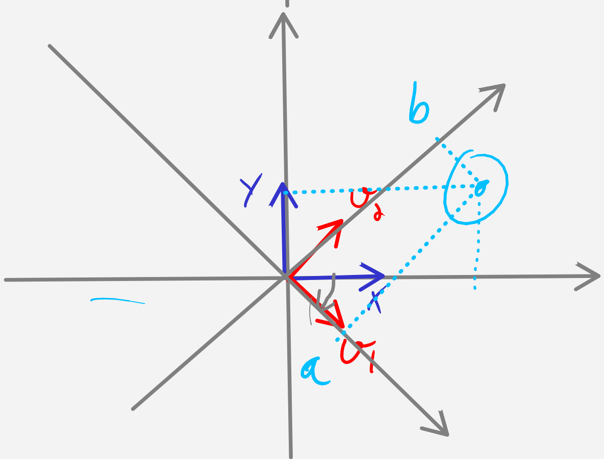
$\|u\| = \sqrt{1+1}$





$\rightarrow v_2 = x(1, 1)$ Normalizado:

$v_2 = \frac{1}{\sqrt{2}}(1, 1)$



$\lambda_1 a^2 + \lambda_2 b^2 = f$

$5a^2 + 3b^2 = f$

$$\frac{a^2}{\left(f/\sqrt{5}\right)^2} + \frac{b^2}{\left(f/\sqrt{3}\right)^2} = 1$$

$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 1 \qquad \alpha = \frac{f}{\sqrt{5}} < \beta = \frac{f}{\sqrt{3}}$$

Base ortonormal:

$$B = \{ v_1, v_2 \} = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$$

Matrizes mudança de base:

Para obter:

$$\begin{bmatrix} a & b \end{bmatrix}_B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + e = 0$$

$$\begin{bmatrix} a \\ b \end{bmatrix}_B = \begin{bmatrix} I \end{bmatrix}_B^C \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a & b \end{bmatrix}_B = \left(\begin{bmatrix} a & b \end{bmatrix}_B \right)^t = \left(\begin{bmatrix} I \end{bmatrix}_B^C \begin{bmatrix} x \\ y \end{bmatrix} \right)^t = \begin{bmatrix} x & y \end{bmatrix}^t \left(\begin{bmatrix} I \end{bmatrix}_B^C \right)^t$$

B e C são ortonormais. Então $[I]_C^B$ é matriz ortogonal:

$$\left([I]_B^C \right)^t = \left([I]_B^C \right)^{-1} = [I]_C^B .$$

$$\begin{bmatrix} a & b \end{bmatrix}_B = \begin{bmatrix} x & y \end{bmatrix} [I]_C^B$$

$$\begin{bmatrix} a & b \end{bmatrix}_B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + e = 0$$

$$\begin{bmatrix} x & y \end{bmatrix}_C \underbrace{\begin{bmatrix} I \end{bmatrix}_C^B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_B \begin{bmatrix} I \end{bmatrix}_B^C}_{\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix}_C + \underbrace{\begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} I \end{bmatrix}_B^C}_{\begin{bmatrix} -8 & -2 \end{bmatrix}} \begin{bmatrix} x \\ y \end{bmatrix} + e = 0.$$

Matriz mudança de base:

$$B = \{ v_1, v_2 \} = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$$

$$\begin{bmatrix} I \end{bmatrix}_C^B = \begin{bmatrix} [v_1]_C & [v_2]_C \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} I \end{bmatrix}_B^C = \left(\begin{bmatrix} I \end{bmatrix}_C^B \right)^{-1} = \left(\begin{bmatrix} I \end{bmatrix}_C^B \right)^t = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \end{bmatrix}_B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + \underbrace{\begin{bmatrix} D & E \end{bmatrix}_C \begin{bmatrix} I \end{bmatrix}_C^B} \begin{bmatrix} a \\ b \end{bmatrix} + F = 0$$

VGA

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} -8 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} a \\ b \end{bmatrix} + F = 0$$

$$5a^2 + 3b^2 + \frac{1}{\sqrt{2}}\left(\begin{bmatrix} -8 & -2 \end{bmatrix} \begin{bmatrix} a+b \\ -a+b \end{bmatrix}\right) + 3 = 0$$

$$5a^2 + 3b^2 + \frac{1}{\sqrt{2}}\left(-8a - 8b + 2a - 2b\right) + 3 = 0$$

$$5a^2 + 3b^2 - \frac{6}{\sqrt{2}}a - \frac{10}{\sqrt{2}}b + 3 = 0$$

$\boxed{\text{Completar quadrados!}}$ exercício (!)

$$5a^2 - \frac{6}{\sqrt{2}}a + k = A(a - a_0)^2 = A(a^2 + 2a a_0 + a_0^2)$$

$$5\left(a^2 - \frac{6}{5\sqrt{2}}a + \left(\frac{3}{5\sqrt{2}}\right)^2\right) = 5(a - a_0)^2$$

$$a_0 = \frac{3}{5\sqrt{2}} //$$

$$3b^2 - \frac{10}{\sqrt{2}}b + r = B(b - b_0)^2$$

$$3\left(b^2 - \underbrace{\frac{10}{3\sqrt{2}}}_{2b_0} b + \left(\frac{5}{3\sqrt{2}}\right)^2\right) = 3(b - b_0)^2$$

$$b_0 = \frac{+5}{3\sqrt{2}}.$$

$$5\underbrace{\left(a - \frac{3}{5\sqrt{2}}\right)^2}_{\tilde{a}^2} + 3\underbrace{\left(b - \frac{5}{3\sqrt{2}}\right)^2}_{\tilde{b}^2} + 3 - b_0^2 - a_0^2 = 0$$

$$5\tilde{a}^2 + 3\tilde{b}^2 + 3 - \left(\frac{5}{3\sqrt{2}}\right)^2 - \left(\frac{3}{5\sqrt{2}}\right)^2 = 0$$

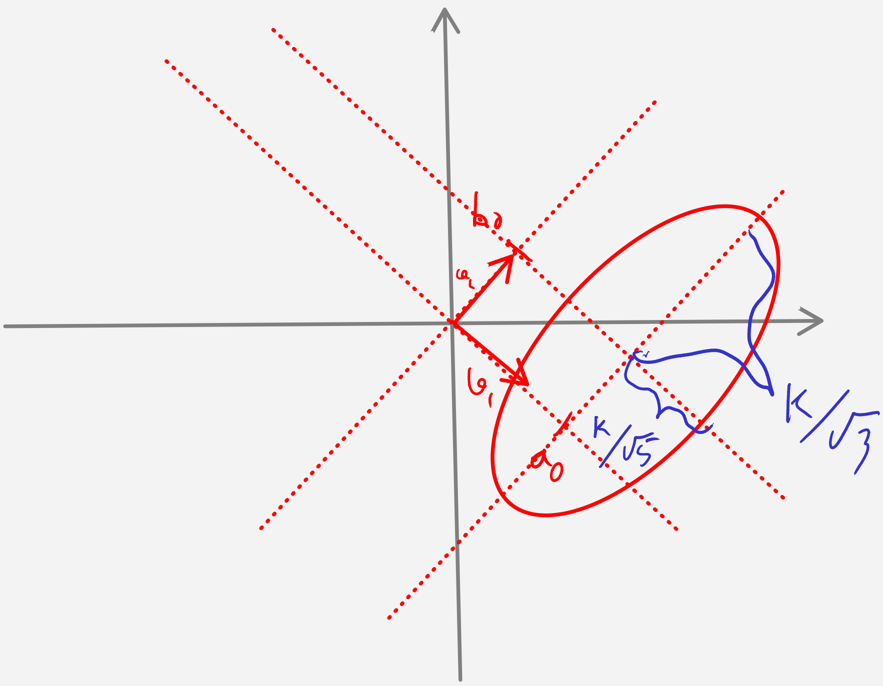
$$5\tilde{a}^2 + 3\tilde{b}^2 + 3 - \frac{25}{18} - \frac{9}{50} = 0$$

$$5\tilde{a}^2 + 3\tilde{b}^2 + 3 - \frac{353}{225}$$

$$5\tilde{a}^2 + 3\tilde{b}^2 = \frac{353}{225} - 3 = -2{,}24$$ ([illegible]?)

$$a_0 = \frac{5}{3\sqrt{2}} \approx 1{,}17$$

$$b_0 = \frac{3}{5\sqrt{2}} \approx 0{,}42$$



$$5\tilde{a}^2 + 3\tilde{b}^2 = K$$

$$\frac{\tilde{a}^2}{\left(K/\sqrt{5}\right)^2} + \frac{\tilde{b}^2}{\left(K/\sqrt{3}\right)^2} = 1$$