Notas de aula ECT2202 T04 2021-04-22 Aula 23 — Diagonalização de f. quadráticas

Forma quadrática

$q: \mathbb{R}^n \to \mathbb{R}$ e

$q(v) = f(v, v)$, onde $f: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ é uma forma bilinear simétrica

Matriz de f na base $B = \{v_1, \dots, v_n\}$

$$[f] = \begin{bmatrix} f(v_1, v_1) & \cdots & f(v_1, v_n) \\ \vdots & \ddots & \vdots \\ f(v_n, v_1) & \cdots & f(v_n, v_n) \end{bmatrix}$$

Produto cartesiano:

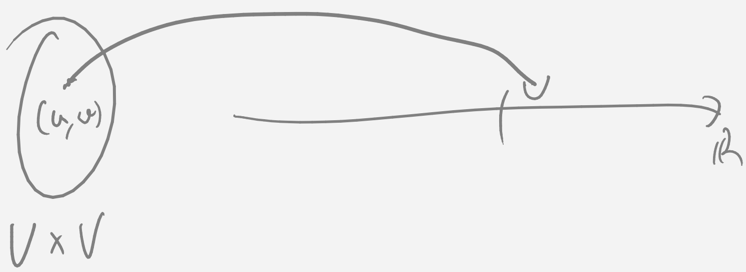
$v \in V$
$w \in W$ o par $(v, w) \in V \times W$

Note: $x \in \mathbb{R}$, $y \in \mathbb{R}$ $\Rightarrow$ $(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$f(u, v)$ é bilinear:

$f(\alpha u + w, v) = \alpha f(u, v) + f(w, v)$

$f(u, \alpha + w) = \alpha f(u, v) + f(u, w)$



Prod. interno: forma bilinear simétrica

$$p(u,v) = \langle u, v \rangle$$

$$p(u,v) = p(v,u)$$

Matriz da forma bilinear simétrica.

$$[f]_B = \begin{bmatrix} f(v_1, v_1) & \cdots & f(v_1, v_n) \\ \vdots & \ddots & \vdots \\ f(v_n, v_1) & \cdots & f(v_n, v_n) \end{bmatrix}$$

Na base $B = \{v_1, \dots, v_n\}$

$$f(u,v) = [u]_B^t \, [f]_B \, [v]_B$$

$$f: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$$

base canônica

$$u = (x_1, y_1) \Rightarrow [u] = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$v = (x_2, y_2) \Rightarrow [v] = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$f(u,v) = \begin{bmatrix} x_1 & y_1 \end{bmatrix} [f] \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$[f] = \begin{bmatrix} f((1,0),(1,0)) & f((1,0),(0,1)) \\ f((0,1),(1,0)) & f((0,1),(0,1)) \end{bmatrix}$$

$$f(u,v) = x_1 x_2 f((1,0),(1,0)) + x_1 y_2 f((1,0),(0,1)) + y_1 x_2 f((0,1),(1,0)) + y_1 y_2 f((0,1),(0,1))$$

Como f é simétrica:

$$f((0,1),(1,0)) = f((1,0),(0,1))$$

Forma quadrática no $\mathbb{R}^2$

$$q(u) = q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \underbrace{[q]}_{2\times 2} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[q] = \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \qquad [(x,y)]\begin{bmatrix} x \\ y \end{bmatrix}$$

$$q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \beta & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \alpha x + \beta y \\ \beta x + \delta y \end{bmatrix}$$

$$= x(\alpha x + \beta y) + y(\beta x + \delta y)$$

$$= \alpha x^2 + \beta x y + \beta y x + \delta y^2$$

$$q(x,y) = \alpha x^2 + 2\beta x y + \delta y^2$$

Sentido contrário:

dada a expressão q(x,y), obter a matriz.

$$q(x,y) = Ax^2 + Bxy + Cy^2$$

A matriz de q será

$$[q] = \begin{array}{c|c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} A & B/2 \\ B/2 & C \end{bmatrix} \end{array}$$ simétrica.

Em $\mathbb{R}^3$:

$$q(x,y,z) = \underline{Ax^2} + \underline{By^2} + \underline{Cz^2} + Dxy + Exz + Fyz$$

$$[q] = \begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} A & D/2 & E/2 \\ D/2 & B & F/2 \\ E/2 & F/2 & C \end{bmatrix} \end{array}$$ ← base canônica

$$[(x,y,z)]_C = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Base B tal que
ortonormal

$$[q]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$q(x,y,z) = \lambda_1 a^2 + \lambda_2 b^2 + \lambda_3 c^2$$

$$[(x,y,z)]_B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

1) Obtenha uma base em que a forma quadrática abaixo seja escrita como soma de quadrados

$$q(x, y) = x^2 - 10xy + y^2$$

$q(x,y) = \lambda_1 a^2 + \lambda_2 b^2$

Algoritmo:
1) Escrever a matriz [q]
2) Obter os autovalores λ de [q]
3) Para cada autovalor λ obter os autovetores L.I. normalizados.
3a) Se houver mais de um autovetor L.I. associado a um autovalor λ, ORTONORMALIZAR o conjunto (Gram-Schmidt!)
4) Normalizar os autovbetores.
5) A matriz diagonal será a matriz dos autovalores.

1) $q(x,y) = x^2 - 10xy + y^2$

Na base canônica:

$$[q] = \begin{bmatrix} 1 & -10/2 \\ -10/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$$

2) Autovalores:

$$\det([q] - \lambda \mathbb{1}) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{bmatrix}\right) = 0$$

$$(1-\lambda)^2 - 25 = 0$$

$$\lambda^2 - 2\lambda + 1 - 25 = 0$$

$$\lambda^2 - 2\lambda - 24 = 0$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(-24)}}{2}$$

$$= \frac{2 \pm \sqrt{4 + 96}}{2} = \frac{2 \pm \sqrt{100}}{2}$$

$$\lambda = 1 \pm 5.$$

$$\lambda_1 = 6 \quad , \quad \lambda_2 = -4.$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ v_1 & \perp & v_2 \end{array}$$

3) Para cada autovalor obter os autovetores correspondentes

Para $\lambda = 6$ Eq. de autovetor:

$$[f][v] = \lambda [v]$$

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x - 5y = 6x \\ -5x + y = 6y \end{cases} \Rightarrow \begin{array}{l} x - 6x - 5y = 0 \\ -5x - 5y = 0 \\ x = -y \end{array}$$

$$v_1 = x(1,-1)$$

Normalizando: $u_1 = \frac{(1,-1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1,-1)$ //

Para $\lambda = -1$

$$\begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x - 5y = -4x \\ -5x + y = -4y \end{cases} \Rightarrow \begin{array}{l} x + 4x - 5y = 0 \\ 5x - 5y = 0 \\ x = y \end{array}$$

$v_1 = x(1,1)$

Normalizando:

$u_1 = \frac{1}{\sqrt{2}}(1,1)$

Na base $B = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$

a matriz será

$$[q]_B = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

Na base B: $[(x,y)]_B = \begin{bmatrix} a \\ b \end{bmatrix}$

A matriz mudança de base:

$$[I]_C^B = \begin{bmatrix} [u_1]_C & [u_2]_C \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_C = [I]^B_C \begin{bmatrix} a \\ b \end{bmatrix}_B$$

$$[I]^C_B = \left([I]^B_C\right)^{-1} = \left([I]^B_C\right)^t$$

$$\left.\begin{array}{l} B \text{ é ortonormal} \\ C \text{ '' } \quad \text{''} \end{array}\right\} \Rightarrow [I]^B_C \text{ é ortogonal: } \left([I]^B_C\right)^{-1} = \left([I]^B_C\right)^t$$

$$[I]^C_B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$[I]^C_B [I]^B_C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \checkmark$$

$$[q]_B = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$$

$$B = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$$

$$[(x,y)]_B = \begin{bmatrix} a \\ b \end{bmatrix}_B = [I]^C_B \begin{bmatrix} x \\ y \end{bmatrix}_C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x-y \\ x+y \end{bmatrix}$$

$$q(x,y) = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

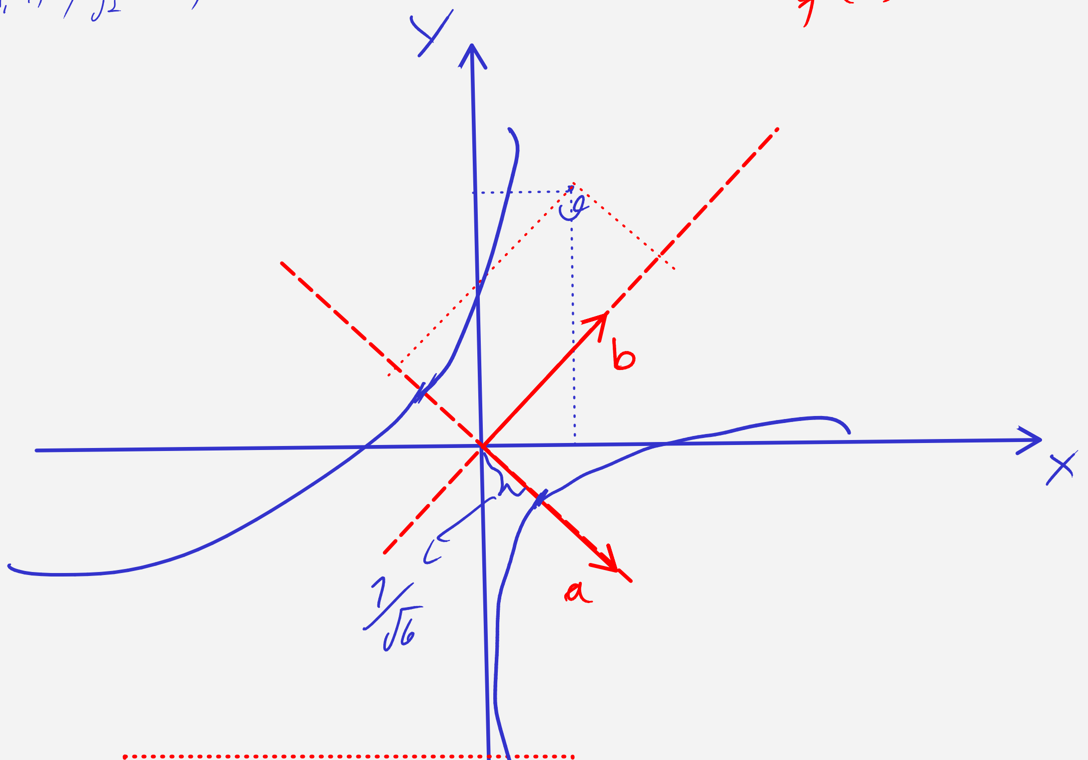
$$q(v) = 6a^2 - 4b^2$$

$$= 6\left(\frac{1}{\sqrt{2}}(x-y)\right)^2 - 4\left(\frac{1}{\sqrt{2}}(x+y)\right)^2$$

Exercicio: verifique que resulta no q(x,y) original

$B = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$

$q(v) = 6a^2 - 4b^2$



$$q(a,b) = \boxed{6a^2 - 4b^2 = 1}$$

Hipérbole

$$q(x,y) = x^2 - 10xy + y^2$$