Notas de aula ECT2202 T04 2021-04-27 Aula 24 — Cônicas e quádricas

Identifique a curva abaixo e a direção de seus eixos principais.

$$4x^2 + 4y^2 - 2xy - 8x - 2y + 3 = 0$$

Equação quadrática no $\mathbb{R}^2$

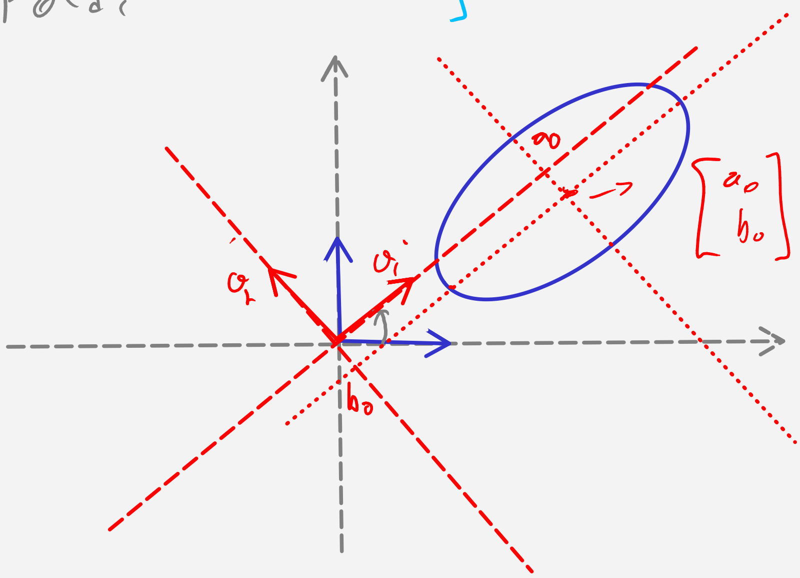
Gráfico:
- Elipse (ou caso degenerado: ponto ou cj. vazio) ← $\lambda_1\lambda_2 > 0$
- Hipérbole (ou caso degenerado: retas concorrentes) ← $\lambda_1\lambda_2 < 0$
- Parábola (ou caso degenerado retas paralelas) ← $\lambda_1\lambda_2 = 0$

$$A x^2 + B xy + C y^2 + D x + E y + F = 0$$

$$\underbrace{4x^2 - 2xy + 4y^2}_{Q} \underbrace{-8x - 2y}_{L} + 3 = 0$$

simétrica

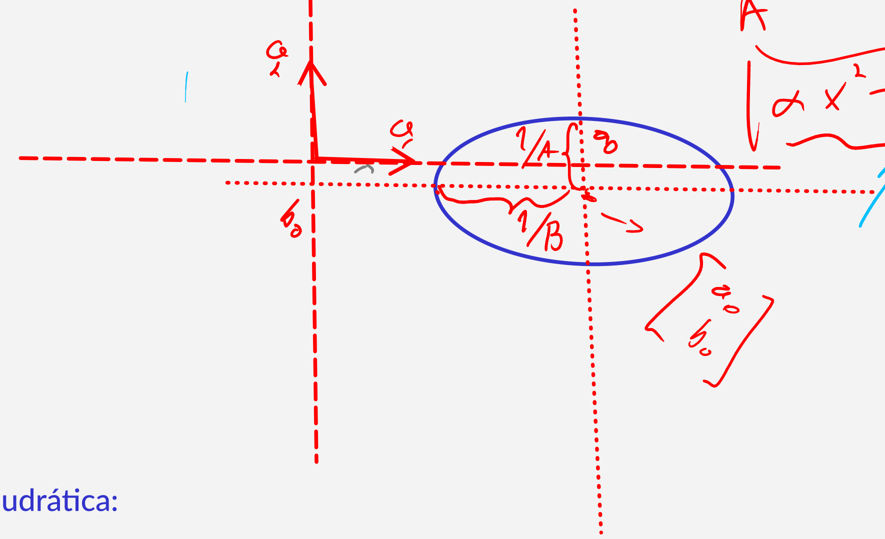
$$\underbrace{\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{rotação}} + \underbrace{\begin{bmatrix} -8 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{\text{translação}} + 3 = 0$$



$Ax^2 + By^2 + Cx + Dy = E$ ← VGA

$\frac{(a-a_0)^2}{A^2} + \frac{(b-b_0)^2}{B^2} = 1$

$\alpha x^2 + \beta y^2 + \gamma x + \delta y = \varphi$



① Diagonalizar a forma qudrática:

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}$$

1) Autovalores: pol. característico.

$$P_Q(\lambda) = \det \begin{bmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 8\lambda + 16 - 1 = 0$$

$$\lambda^2 - 8\lambda + 15 = 0 \Rightarrow \begin{cases} \lambda_1 + \lambda_2 = 8 \\ \lambda_1 \lambda_2 = 15 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 5 \\ \lambda_2 = 3 \end{cases}$$

$\lambda_1 \lambda_2 > 0$, mesmo sinal

Elipse (ou caso degenerado)

Eixos da elipse: na direção dos autovetores da forma quadrática

Autovetores:

$\lambda_1 = 5$: $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix}$

Simétrica

Existe base ORTONORMAL em que a matriz desta forma quadrática é diagonal.

A matriz mudança de base será uma matriz ortogonal: $M^{-1} = M^T$

$$\begin{cases} 4x - y = 5x \\ -x + 4y = 5y \end{cases}$$

$$\begin{cases} -x - y = 0 \\ -x - y = 0 \end{cases} \Rightarrow x = -y$$

$v_1 = x(1,-1)$

preciso que $\|v_1\| = 1 \Rightarrow x = \dfrac{1}{\sqrt{(1)^2 + (-1)^2}}$

$$v_1 = \frac{1}{\sqrt{2}}(1,-1)$$

Preciso $\langle v_1, v_2 \rangle = 0$

$\|v_2\| = 1 \Rightarrow v_2 = \dfrac{1}{\sqrt{2}}(1,1)$

Segundo autovetor

$\lambda_2 = 3$ $\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$

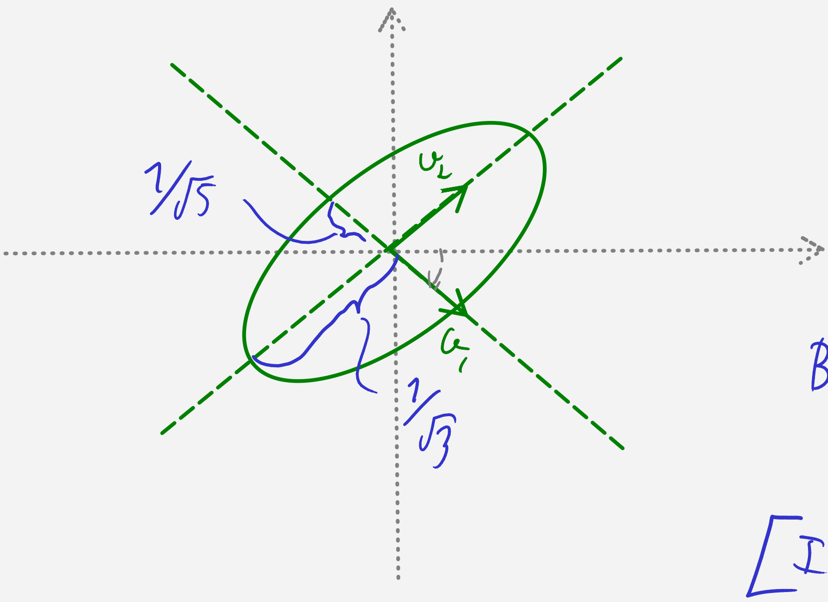
$$\begin{cases} 4x - y = 3x \\ -x + 4y = 3y \end{cases} \Rightarrow \begin{cases} x - y = 0 \Rightarrow x = y \\ -x + y = 0 \end{cases}$$

$$v_2 = x(1,1)$$

$\|v_2\| = 1 \Rightarrow x = \dfrac{1}{\sqrt{1^2 + 1^2}} \Rightarrow v_2 = \dfrac{1}{\sqrt{2}}(1,1).$

$\to \lambda_1 = 5 \quad , \lambda_2 = 3 \quad \Rightarrow$ Elipse (ou deg).

$v_1 = \frac{1}{\sqrt{2}}(1,-1) \quad v_2 = \frac{1}{\sqrt{2}}(1,1)$



Base ortonormal

$B = \{ v_1, v_2 \}$

$B = \left\{ \frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1) \right\}$

$$[I]_C^B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Tenho:

$$[x \;\; y]_C \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}_C \begin{bmatrix} x \\ y \end{bmatrix}_C + [-8 \;\; -2]_C \begin{bmatrix} x \\ y \end{bmatrix}_C + 3 = 0$$

quero

$$[a \;\; b]_B \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + [d \;\; e]_B \begin{bmatrix} a \\ b \end{bmatrix}_B + 3 = 0$$

$$[-8 \;\; -2]_C \begin{bmatrix} x \\ y \end{bmatrix}_C = [d \;\; e]_B \begin{bmatrix} a \\ b \end{bmatrix}_B$$

$$\begin{bmatrix} x \\ y \end{bmatrix}_C = [I]_C^B \begin{bmatrix} a \\ b \end{bmatrix}_B$$

$$\left(\underbrace{\begin{bmatrix} -8 & -2 \end{bmatrix}_C [I]_C^B}_{[d \;\; e]}\right)\begin{bmatrix} a \\ b \end{bmatrix}_B = \begin{bmatrix} d & e \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$$

$$[I]_C^B = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} d & e \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -8 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -8+2 & -8-2 \end{bmatrix}$$

$$\begin{bmatrix} d & e \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -6 & -10 \end{bmatrix}.$$

$$\begin{bmatrix} a & b \end{bmatrix}_B \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + \begin{bmatrix} \frac{-6}{\sqrt{2}} & \frac{-10}{\sqrt{2}} \end{bmatrix}_B \begin{bmatrix} a \\ b \end{bmatrix}_B + 3 = 0$$

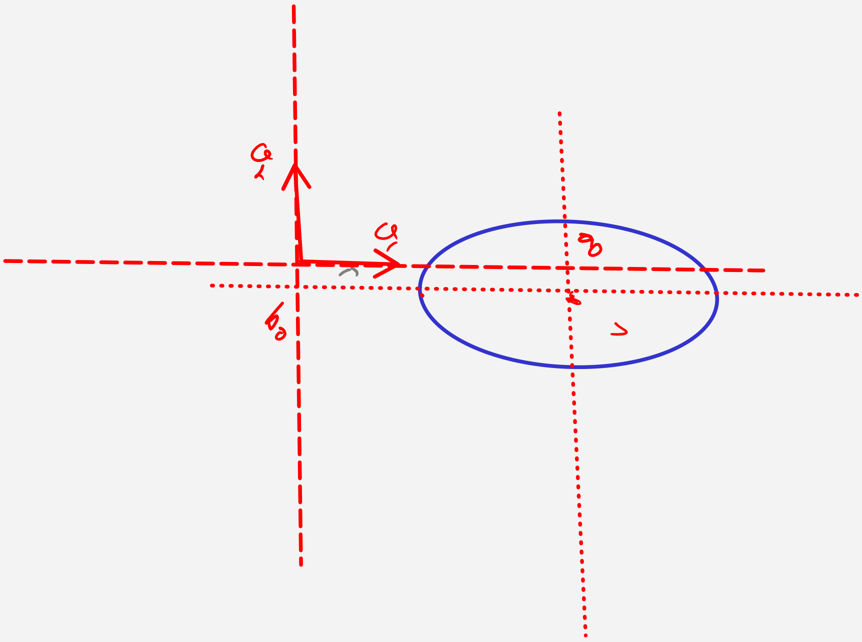
$$\boxed{5a^2 + 3b^2 - \frac{6}{\sqrt{2}}a - \frac{10}{\sqrt{2}}b + 3 = 0.}$$ VGA.

Completar quadrados

$$5(\tilde{a} - a_0)^2 + 3(\tilde{b} - b_0)^2 = k$$ se $k=0$, ponto

$$\frac{(\tilde{a} - a_0)^2}{\left(\sqrt{\frac{k}{5}}\right)^2} + \frac{(\tilde{b} - b_0)^2}{\left(\sqrt{\frac{k}{3}}\right)^2} = 1 \Leftrightarrow$$ se $k > 0$

se $k < 0$: vazio.



$$5a^2+3b^2-\frac{6}{\sqrt{2}}a-\frac{10}{\sqrt{2}}b+3=0.$$

$$\left(5a^2-\frac{6}{\sqrt{2}}a+a_0^2\right)+\underbrace{\left(3b^2-\frac{10}{\sqrt{2}}b+b_0^2\right)}-a_0^2-b_0^2+3=0$$

$$5(a-a_0)^2 \quad + \quad 3(b-b_0)^2 \qquad -a_0^2-b_0^2+3=0$$

$$\left\{\begin{array}{l} 5a^2-\frac{6}{\sqrt{2}}a+a_0^2 \Leftrightarrow 5(a-a_0)^2 \\ 5\left(a^2-\frac{6}{5\sqrt{2}}a+a_0^2\right)=5\left(a^2-2a_0a+a_0^2\right) \\ 2a_0=\frac{6}{5\sqrt{2}} \Rightarrow \boxed{a_0=\frac{3}{5\sqrt{2}}} \end{array}\right.$$

$$\left\{\begin{array}{l} 3b^2-\frac{10}{\sqrt{2}}b+b_0^2=3(b-b_0)^2 \\ 3\left(b^2-\frac{10}{3\sqrt{2}}b+b_0^2\right)=3\left(b^2-2b_0b+b_0^2\right) \\ 2b_0=\frac{10}{3\sqrt{2}} \Rightarrow \boxed{b_0=\frac{5}{3\sqrt{2}}} \end{array}\right.$$

$$5\underbrace{\left(a-\frac{3}{5\sqrt{2}}\right)}_{\bar{a}}^{2}+3\underbrace{\left(b-\frac{5}{3\sqrt{2}}\right)}_{\bar{b}}^{2}\underbrace{-\left(\frac{3}{5\sqrt{2}}\right)^{2}-\left(\frac{5}{3\sqrt{2}}\right)^{2}+3}_{-k}=0$$

$$5\bar{a}^{2}+3\bar{b}^{2}=k$$

$$\frac{\bar{a}^{2}}{\left(\sqrt{k/5}\right)^{2}}+\frac{\bar{b}^{2}}{\left(\sqrt{k/5}\right)^{2}}=1 \quad //$$

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$$\boxed{\frac{\bar{a}^2}{A} + \frac{\bar{b}^2}{B} = 1}$$

$A = 1/\lambda_1$

$B = 1/\lambda_2$

$$\frac{\bar{a}^2}{1/\lambda_1} + \frac{\bar{b}^2}{1/\lambda_2} = 1$$

$\lambda_1 > 0, \lambda_2 > 0$: elipse

$$\boxed{\lambda_1 > 0, \lambda_2 < 0}$$

$$\boxed{\frac{\bar{a}^2}{1/\lambda_1} - \frac{\bar{b}^2}{|1/\lambda_2|} = 1}$$ Hipérbole.

$|1/\lambda_2| > 0$

Eq. Hipérbole: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$b^2 > 0$

$\lambda_1 \neq 0 \quad \lambda_2 = 0$

$\lambda_1 a^2 + a + b = 1$