Notas de aula ECT2202 T04 2021-04-15 Aulas 20 e 21 — Diagonalização

Considere o operador $F : \mathbb{R}^3 \to \mathbb{R}^3$ tal que $[F] = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ em relação à base canônica. A base em que sua matriz é diagonal é:

$B = \{ v_1, v_2, v_3 \}$ tal que

(ordenada)

$$[F]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} [F(v_1)]_B & [F(v_2)]_B & [F(v_3)]_B \end{bmatrix}$$

$$[F(v_1)]_B = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow F(v_1) = \lambda_1 v_1 + 0 v_2 + 0 v_3$$

Analogamente: $F(v_2) = \lambda_2 v_2$

$F(v_3) = \lambda_3 v_3$

Ou seja: v_1, v_2, v_3 são autovetores de F

Obter autovalores e autovteores de F

1) Autovalores: polinômio Característico.

$$[F] = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$p_F(\lambda) = \det([F] - \lambda \mathbb{1}) = \det \begin{pmatrix} 2-\lambda & 0 & 4 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

cofatores:

$$\begin{vmatrix} 2-\lambda & 0 & 4 \\ 1 & -1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} + 4\begin{vmatrix} 1 & -1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (2-\lambda)(2-\lambda)\underline{(-1-\lambda)} - 4\underline{(-1-\lambda)}$$

$$= (-1-\lambda)\left[(2-\lambda)^2 - 4\right] = 0$$

$$(-1-\lambda) = 0 \Rightarrow \boxed{\lambda = -1}$$

ou $(2-\lambda)^2 - 4 = 0$

$$\lambda^2 - 4\lambda + 4 - 4 = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0 \Rightarrow \begin{cases} \boxed{\lambda = 0} \\ \text{ou} \\ \boxed{\lambda = 4} \end{cases}$$

3 autovalores distintos.

Condição SUFICIENTE para que F seja diagonalizável: $\dim(\mathbb{R}^3) = 3$

3 subespaços L.I.

$$\left\{ \underset{v_1}{V(\lambda_1 = 4)}^{\dim 1},\ \underset{v_2}{V(\lambda_2 = 0)}^{\dim 1},\ \boxed{\underset{v_3}{V(\lambda_3 = -1)}}^{\dim 1} \right. \text{ L.I.}$$

3 vetores L.I. $\Rightarrow$ BASE de $\mathbb{R}^3$.

Base $B = \{v_1, v_2, v_3\}$ de autovetores de T.

2) Obter os autovetores associados a cada autovalor

$\lambda_1 = 4$:

$$T(v_1) = \lambda_1 v_1$$

$$[T][v_1] = \lambda_1 [v_1]$$

$$[v_1] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 4 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} 2x + 4z = 4x \\ x - y = 4y \\ x + 2z = 4z \end{cases}$$

$$\begin{cases} -2x + 4z = 0 \\ x - 5y = 0 \Rightarrow 5y = x = 2z \\ \qquad\qquad\qquad y = \frac{2}{5} z \\ x - 2z = 0 \Rightarrow 2z = x \end{cases}$$

$v_1 = z\left(2, \frac{2}{5}, 1\right)$ escolho $z = 5$:

$V(\lambda_1 = 4) = [(10, 2, 5)] \in$

$v_1 = ((0, 2, 5))$.

$\lambda = 0$ $\quad T(v_2) = 0\, v_2 \Rightarrow T(v_2) = 0$

$(\ker(F))$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} 2x + 4z = 0 \\ x - y = 0 \\ x + 2z = 0 \end{cases}$$

$\Rightarrow x = y = -2z$

$\Rightarrow x = -2z$

z . var. livre

$v_2 = z(-2, -2, 1)$

$V(\lambda = 0) = [(-2, -2, 1)]$

$v_2 = (-2, -2, 1)$ //

$\lambda_3 = -1$ $\quad \boxed{T(v_3) = \lambda_3 v_3}$

$[T][v_3] = \lambda_3 [v_3]$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} - (-1) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{cases} 2x + 4z = -x \\ x - y = -y \\ x + 2z = -z \end{cases} \Rightarrow$$

$$\begin{cases} 2x + x + 4z = 0 \\ x - y + y = 0 \\ x + 2z + z = 0 \end{cases}$$

$$\rightarrow \begin{cases} 3x + 4z = 0 \\ x = 0 \\ x + 3z = 0 \end{cases} \quad \begin{array}{l} \Rightarrow x = 0 \\ \\ \Rightarrow z = 0 \end{array}$$

y : livre.

$$v_3 = y\,(0,1,0)$$

$$V(\lambda_3 = -1) = [(0,1,0)].$$

$$v_3 = (0,1,0) \;//$$

$$B = \{\underset{v_1}{(10,2,5)}, \underset{v_2}{(-2,-2,1)}, \underset{v_3}{(0,1,0)}\}.$$

$$[F]_B = \Big[\, [F(v_1)]_B \quad [F(v_2)]_B \quad [F(v_3)]_B \,\Big]$$

$$F(v_1) = \lambda_1 v_1 \qquad F(v_2) = \lambda_2 v_2$$

$$[F(v_1)]_B = \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} \qquad [F(v_2)] = \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \end{bmatrix}$$

$$[F]_B = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{diagonal.} \quad //$$

$$\boxed{[F] = A \qquad [F]_B = D}$$

$$[F(v)]_B = [F]_B [v]_B$$

$$[F(v)]_C = [F]_C [v]_C$$

Matriz mudança de base:

$$\boxed{P = [I]^B_C} = \big[[v_1]_C \ [v_2]_C \ [v_3]_C \big] = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 10 & -2 & 0 \\ 2 & -2 & 1 \\ 5 & 1 & 0 \end{bmatrix} \end{array}$$

$$P^{-1} = \left([I]^B_C\right)^{-1} = [I]^C_B$$

$$[F(v)]_B = [I]^C_B [F(v)]_C$$

$$= [I]^C_B \left([F]_C [v]_C\right)$$

$$[F(v)]_B = [I]^C_B \, [F]_C \, [v]_C$$

$$= [I]^C_B \, [F]_C \, [I]^B_C \, [v]_B$$

$$[F(v)]_B = \underbrace{\left([I]^C_B \, [F]_C \, [I]^B_C \right)}_{\downarrow} [v]_B$$

$$[F(v)]_B = \qquad [F]_B \qquad [v]_B$$

Logo $\quad [F]_B = [I]^C_B \, [F]_C \, [I]^B_C$

$$D = P^{-1} \, A \, P$$

$$P D = A P$$

$$P D P^{-1} = A \longleftarrow$$

$A = [F]_C$ e $D = [F]_B$ são matrizes semelhantes.

Raízes duplas?

Se λ é raiz dupla.

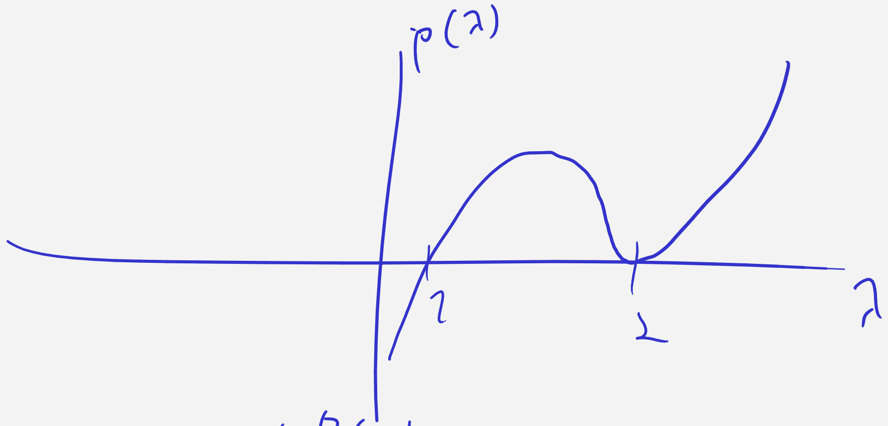
$$\dim(V(\lambda)) \leq 2$$

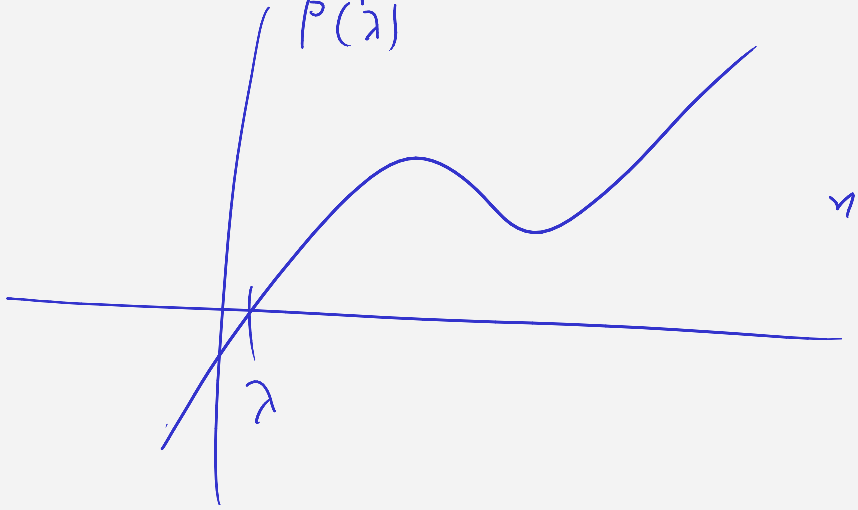
Se $\dim(V(\lambda)) = 1$, então ñ é diag.

$$p_T(\lambda) = (\lambda-2)^2(\lambda-1)$$

$\lambda_1 = 2$ é raiz. $\Rightarrow$ se $\dim(V(\lambda_1)) = 2$. então diag.

$\lambda_2 = 1$ é raiz





Não é diag. no $\mathbb{R}^3$.